# A story of dimension reduction: the generalized pion in the magnetic field

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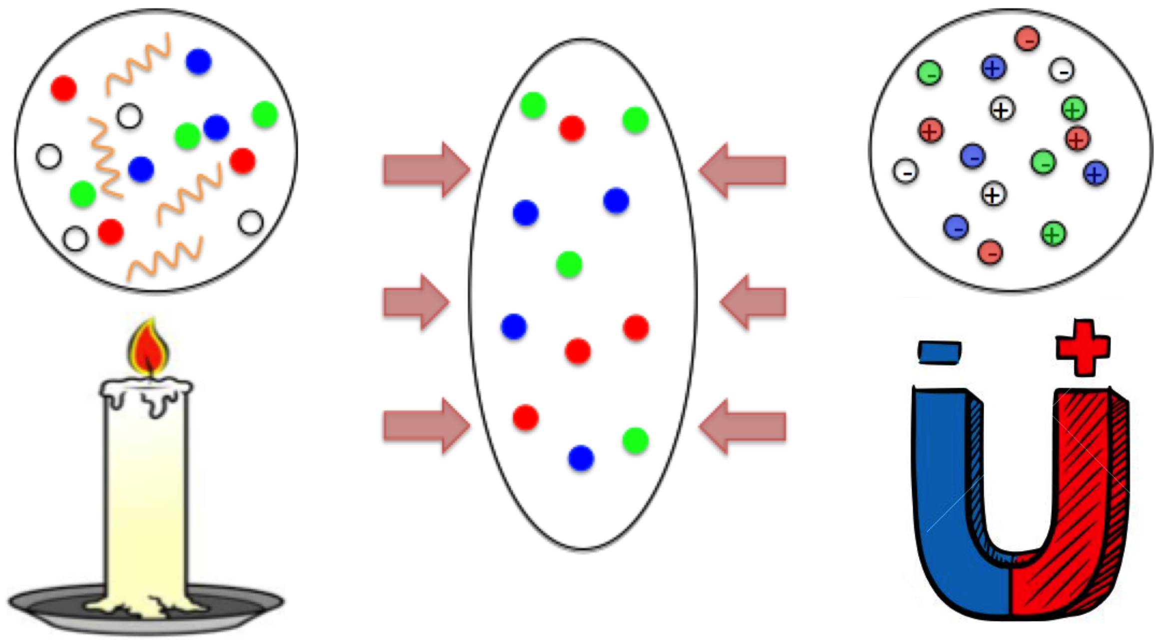
# Outline

- Motivations
  - ✓ Different phases of QCD occur in the universe
    Neutron Stars, Big Bang, Laboratories
  - ✓ Lattice results
- Dimension reduction
- Revisited dimension reduction
- Possible pion superfluid

# Why electromagnetic fields

Different excited freedoms at different environments



strong magnetic field probes physics at short distances $1/\sqrt{eB}$

# Landau levels with AMM

- Summation representation of the fermion propagator in the $B$ field:

$$S(k) = i\exp\left[-\frac{k_\perp^2}{eB}\right]\sum_{n=0}^{\infty}\frac{(-1)^n D_n(eB,k)}{k_0^2-k_3^2-M^2-2neB}$$

and

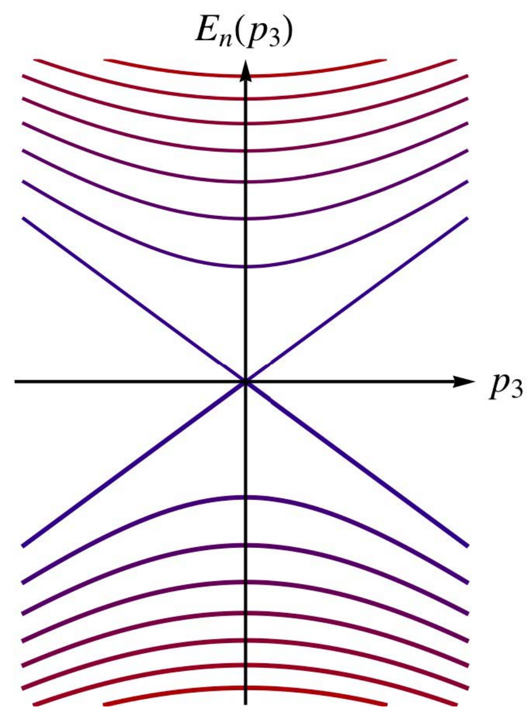
$$\begin{aligned} D_n(q_f,k) &= (\not{k}_{\parallel}+M)\left[P_-L_n - P_+L_{n-1}\right] \\ &\quad +4\not{k}_\perp L_{n-1}^1 \end{aligned}$$

Note: $P_\pm = 1 \pm i\gamma^1\gamma^2\mathrm{sign}(q_f)$, $L_0=1$, $L_{-1}^{(0,1)}=0$.

- Free energy spectrum

$$E_n^{3+1}(k_3) = \pm\sqrt{(2n+2s_3+1)|eB| + k_3^2 + m^2}$$

where $s_3$ is the projection of the spin on the $B$ field and $n = 0, 1, 2, \ldots$ is the orbital quantum number.



# Dimension reduction in the fermion sector

Lowest Landau Level assumption of fermions in 3-direction while $B \longrightarrow \infty$

$$E_0^{3+1}(k_3) = \pm|k_3|$$

$$\tilde{S}^{(0)}(k) = i\ \exp\left(-\frac{\mathbf{k}_\perp^2}{|eB|}\right)\frac{k^0\gamma^0 - k^3\gamma^3 + m}{k_0^2 - k_3^2 - m^2}(1 - i\gamma^1\gamma^2\mathrm{sign}(eB))\ . \qquad (18)$$

This equation clearly demonstrates the $(1+1)$–dimensional character of the LLL dynamics in the infrared region, with $\mathbf{k}_\perp^2 \ll |eB|$. Since at $m^2, k_0^2, k_3^2, \mathbf{k}_\perp^2 \ll |eB|$ the LLL pole dominates in the fermion propagator, one concludes that the dimensional reduction $3+1 \to 1+1$ takes place for the infrared dynamics in a strong (with $|eB| \gg m^2$) magnetic field. It is clear that such a dimensional reduction reflects the fact that the motion of charged particles is restricted in directions perpendicular to the magnetic field.

c.f. (V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Nucl. Phys. B 462, 249)

# Nambu–Jona-Lasino model of QCD

- low energy effective model with four-fermion contact interactions

$$\mathcal{L} = \bar{\psi}\left(\not{D} - m\right)\psi + G_S\left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}\, i\gamma_5\vec{\tau}\psi\right)^2\right]$$

- after the Hubbard Stratonovich transformation, this is equivalent to

$$\mathcal{L}_{eff} = \frac{\sigma^2 + \pi^2}{4G_S} + \ln\det\left(i\not{D} + m_0 - \sigma - i\gamma_5\pi\right)$$

- the following composite fields were introduced

$$\sigma \sim -G_S\left\langle\bar{\psi}\psi\right\rangle, \qquad \pi \sim -G_S\left\langle\bar{\psi}\, i\gamma_5\psi\right\rangle.$$

- at $B = 0$, the free energy:

$$\mathcal{F} = \frac{M^2}{2G_S} + \frac{1}{(4\pi)^2}\left[\frac{\Lambda^4}{2} - 2\Lambda^2 M^2 + \frac{M^4}{2} + M^4\ln\frac{M^2}{\Lambda^2}\right]$$

# The modification of the gap equation with $B$

- gap equation in NJL model

$$M\left[\frac{4\pi^2}{G_S}-\Lambda^2+M^2\ln\frac{M^2}{\Lambda^2}\right]=0$$

- nontrivial solution for

$$G_S>G_{cr}=\frac{4\pi^2}{\Lambda^2}$$

- gap equation for nonzero $B$

$$\frac{4\pi^2}{G_S}-\Lambda^2+M^2\ln\frac{M^2}{\Lambda^2}-|2q_fB|\left[\zeta^{(1,0)}(0,x_f)+x_f-\frac{2x_f-1}{2}\ln x_f\right]=0$$

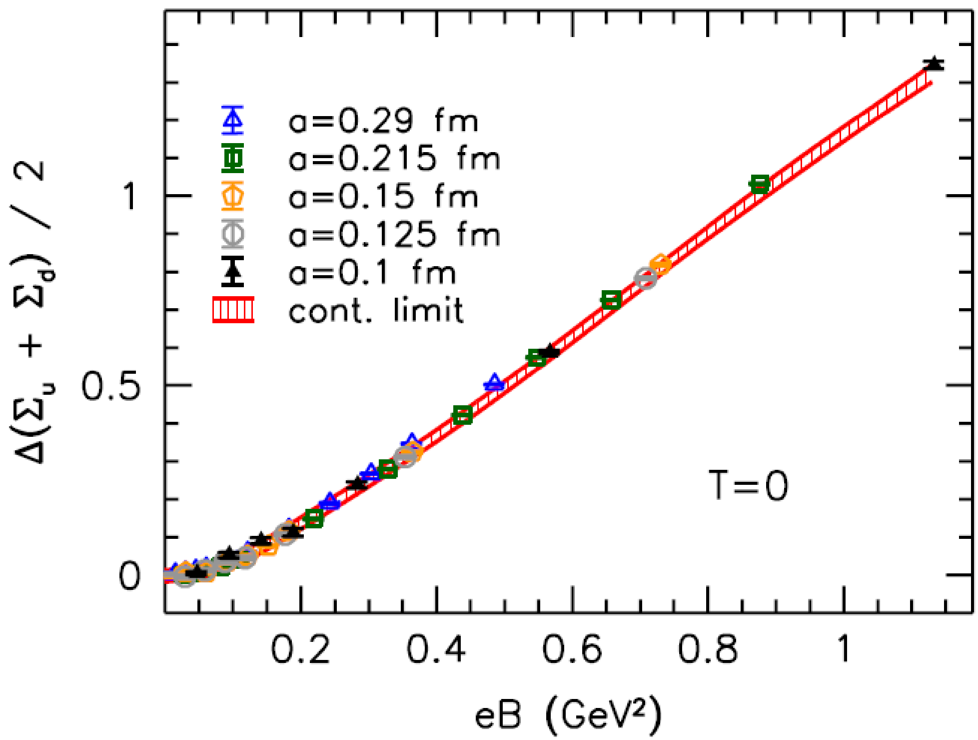
- solution exists for $G_S<G_{cr}$

$$M^2=\frac{|q_fB|}{\pi}\exp\left[-\frac{1}{|q_fB|}\left(\frac{4\pi^2}{G_S}-\Lambda^2\right)\right]$$

The infrared dynamics ($\mathrm{tr}\,\tilde{S}_{LLL}\sim m\ln m^2$) generates a dynamical mass for fermions even at the weakest attractive interactions. c.f. (V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Nucl. Phys. B 462, 249)

# Quark sector: chiral condensate at zero temperature

The LLL dominance can, in particular, be seen in the calculation of the chiral condensate which is linearly increasing as function of $B$




c.f. (Bali et al., JHEP 1202, 044)

# Meson sector: Nambu-Goldstone boson

- There should be massless NG bosons created due to the number of broken-symmetry generators.
- Magnetic field modifies the $SU(2)$ isospin symmetry; global chiral symmetry $U_V(1) \times U_A(1)$ breaks down to $U_{L+R}(1)$.
- Only the neutral pion is a Goldstone mode under $B$-field.
- Another light pseudo-NG boson, the axion may appear in a very strong magnetic field due to the $U_A(1)$.

## Goldstone's theorem in 2-D

There are no Goldstone Bosons in Two Dimensions*

Sidney Coleman**
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey, USA

Received February 1, 1973

This is because if such a spontaneous symmetry breaking occurred, then the corresponding Goldstone bosons, being massless, would have an infrared divergent correlation function.

In turn, the fluctuations are strong enough to destroy the spontaneous symmetry breaking.

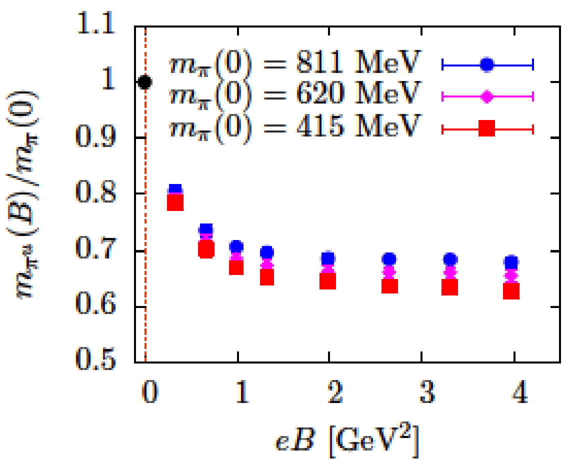
# Mermin-Wagner-Coleman theorem in the magnetic field

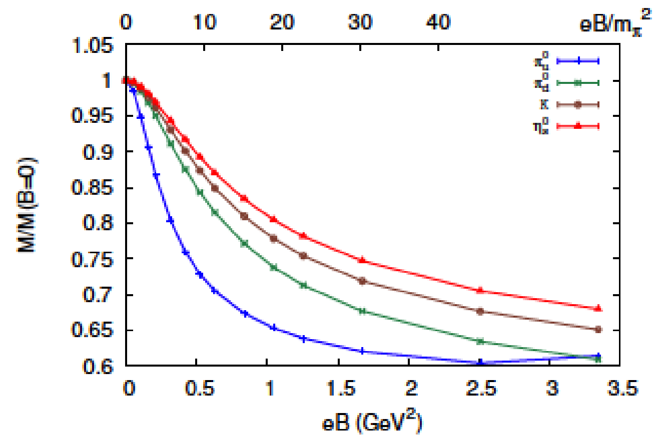
However, the MWC theorem is not applicable to the present problem. The central point is that the condensate $\langle 0|\bar{\psi}\psi|0\rangle$ and the NG modes are **neutral** in this problem and the dimensional reduction in a magnetic field does not affect the dynamics of the center of mass of **neutral** excitations. Indeed, the dimensional reduction $D \rightarrow D-2$ in the fermion propagator, in the infrared region, reflects the fact that the motion of **charged** particles is restricted in the directions perpendicular to the magnetic field. Since there is no such restriction for the motion of the center of mass of neutral excitations, their propagators have $D$–dimensional form in the infrared region (since the structure of excitations is irrelevant at

c.f. (V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Nucl. Phys. B 462, 249)

# Lattice results of neutral pion

Neutral pion monotonouslly decrease as the magnetic field grows and then saturate at a nonzero value




Results for the neutral pion masses. c.f. (Bali, G. et al., Phys. Rev. D 97 (2017): 034505; Ding, H. et al., arXiv:2001.05322)

The modified Gell-Mann-Oakes-Renner (GMOR) relation $m_\pi^2 f_\pi^2 = m_0\langle\bar{\psi}\psi\rangle$. c.f. (A. Sidney, et.al., Phys. Rev. D 93 (2016) 014010)

# Anisotropic four-fermion interactions

one-gluon exchange based four-fermion interactions

$$\mathcal{L}_{int} = g_{\shortparallel}^2 \left(\bar{\psi}\gamma_\mu^{\shortparallel}\psi\right)^2 + g_\perp^2 \left(\bar{\psi}\gamma_\mu^{\perp}\psi\right)^2 .$$

apply the anisotropic Fierz identities c.f. (Ferrer, Efrain J., et al. , Phys. Rev. D 89 (2014), Phys. Rev. D 085034.)

$$\begin{aligned}
\left(\gamma_\mu^{\shortparallel}\right)_{il}\left(\gamma^\mu_{\shortparallel}\right)_{jk} &= \frac{1}{2}(1)_{il}(1)_{jk} + \frac{1}{2}(i\gamma_5)_{il}(i\gamma_5)_{jk} \\
&+\frac{1}{4}\left(\sigma_\perp^{\mu\nu}\right)_{il}\left(\sigma^\perp_{\mu\nu}\right)_{jk} - \frac{1}{2}\left(\sigma_{\shortparallel}^{03}\right)_{il}\left(\sigma^{\shortparallel}_{03}\right)_{jk} + \ldots, \\
\left(\gamma_\mu^{\perp}\right)_{il}\left(\gamma^\mu_{\perp}\right)_{jk} &= \frac{1}{2}(1)_{il}(1)_{jk} + \frac{1}{2}(i\gamma_5)_{il}(i\gamma_5)_{jk} \\
&-\frac{1}{4}\left(\sigma_\perp^{\mu\nu}\right)_{il}\left(\sigma^\perp_{\mu\nu}\right)_{jk} + \frac{1}{2}\left(\sigma_{\shortparallel}^{03}\right)_{il}\left(\sigma^{\shortparallel}_{03}\right)_{jk} + \ldots
\end{aligned}$$

It concludes that the differing $g_{\shortparallel,\perp}$ will manifest themselves by considering four-point coupling in tensor channel and dynamically generate anomalous magnetic moment like dynamically generated mass.

# AMM arising from the tensor channels

$$\mathcal{L}_{\rm int} = G_S \left[ \left( \bar{\psi}\psi \right)^2 + \left( \bar{\psi}\, i\gamma_5 \vec{\tau}\psi \right)^2 \right] + G_T \left[ \left( \bar{\psi}\sigma^{12}\tau_a\psi \right)^2 + \left( \bar{\psi}\, i\gamma_5\sigma^{12}\psi \right)^2 \right]$$

where $G_T \leq G_S$ since $G_S \sim g_{\shortparallel}^2 + g_{\perp}^2$ and $G_S \sim g_{\shortparallel}^2 - g_{\perp}^2$. The transverse $\sigma^{12}$ index is chosen w.r.t. the magnetic field pointing in the $z$-direction. Note: $a = 0, 3$ for $\tau_a$.

$$M \sim -G_S \left\langle \bar{\psi}\psi \right\rangle, \qquad \kappa \sim -G_T \left\langle \bar{\psi}\sigma^{12}\tau_a\psi \right\rangle.$$

The Dirac propagator is in the form of

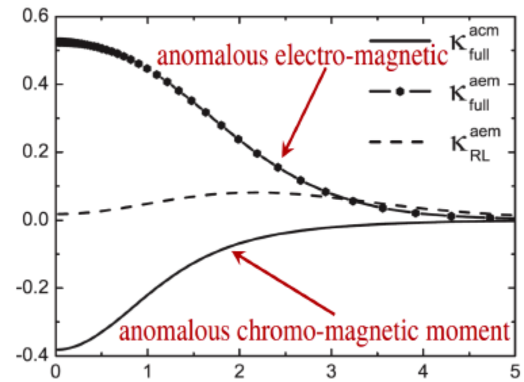
$$\tilde{G}(q_f, k) = \exp\left[ -\frac{\mathbf{k}_\perp^2}{|q_f|B} \right] \sum_{\pm} \sum_{n=0}^{\infty} (-1)^n \frac{\mathcal{D}_n(q_f B, k)\, \Lambda_\pm}{k_{\shortparallel}^2 - 2n|q_f|B - M^2 + \kappa^2 \pm 2|\kappa k_{\shortparallel}|},$$

where $\Lambda_\pm = \frac{1}{2} \pm \frac{\gamma^3\gamma^5 k_0 - \gamma^0\gamma^5 k_3}{2|k_{\shortparallel}|} \mathbf{sign}(\kappa)$ and

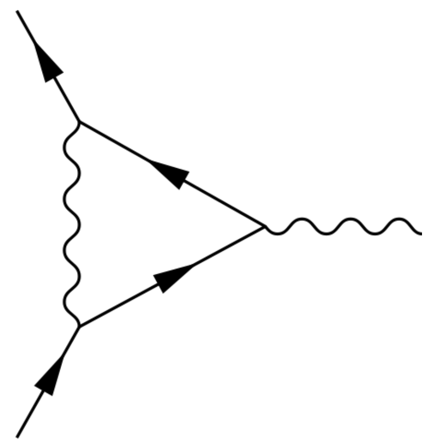
$$\mathcal{D}_n(q_f, k) = \left( \not{k}_{\shortparallel} + M + \kappa \left( q_f \right) \sigma \hat{F} \right) \left[ P_- L_n - P_+ L_{n-1} \right] + 4\not{k}_\perp L_{n-1}^1.$$

# The source and the sign of AMM

- Flavor blind $\sim \tau_0$, coming form the compensation of color interaction c.f. (L. Chang, Y.-X. Liu, C. D. Roberts, PRL 106, (2011) 072001 )




- Charge dependent $\sim q_f$, due to the fluctuations of QED



- Or simply flavor dependent $\sim \mathrm{sign}\,(q_f) = \tau_3$

# Infrared logarithmic enhanced VEVs in the LLL approximation

The applied mean-field approximation is recognizing by the gap equations:

$$\frac{M-m}{2\,iG_S} = \mathrm{Tr}\,G;$$
$$\frac{\kappa}{2\,iG_T} = \mathrm{Tr}\left[\sigma^{12}G\right];$$

$$\mathrm{Tr}\,G(k) = I_1 + I_2, \qquad \mathrm{sign}(\kappa)\,\mathrm{Tr}\left[\sigma^{12}G(k)\right] = I_1 + I_3.$$

Solution exists while $\mathbf{sign}(\kappa) = -\mathbf{sign}(q_f)$ and $\tau_a = -\tau_3$

$$I_1 = N_c \sum_{q_f} \frac{|q_f|B}{8\pi^3} \int \frac{d^2k_{\shortparallel}}{|k_{\shortparallel}| - M + \kappa} \sim (M-\kappa) \int \frac{d^2k_{\shortparallel}}{k_{\shortparallel}^2 - (M-\kappa)^2};$$

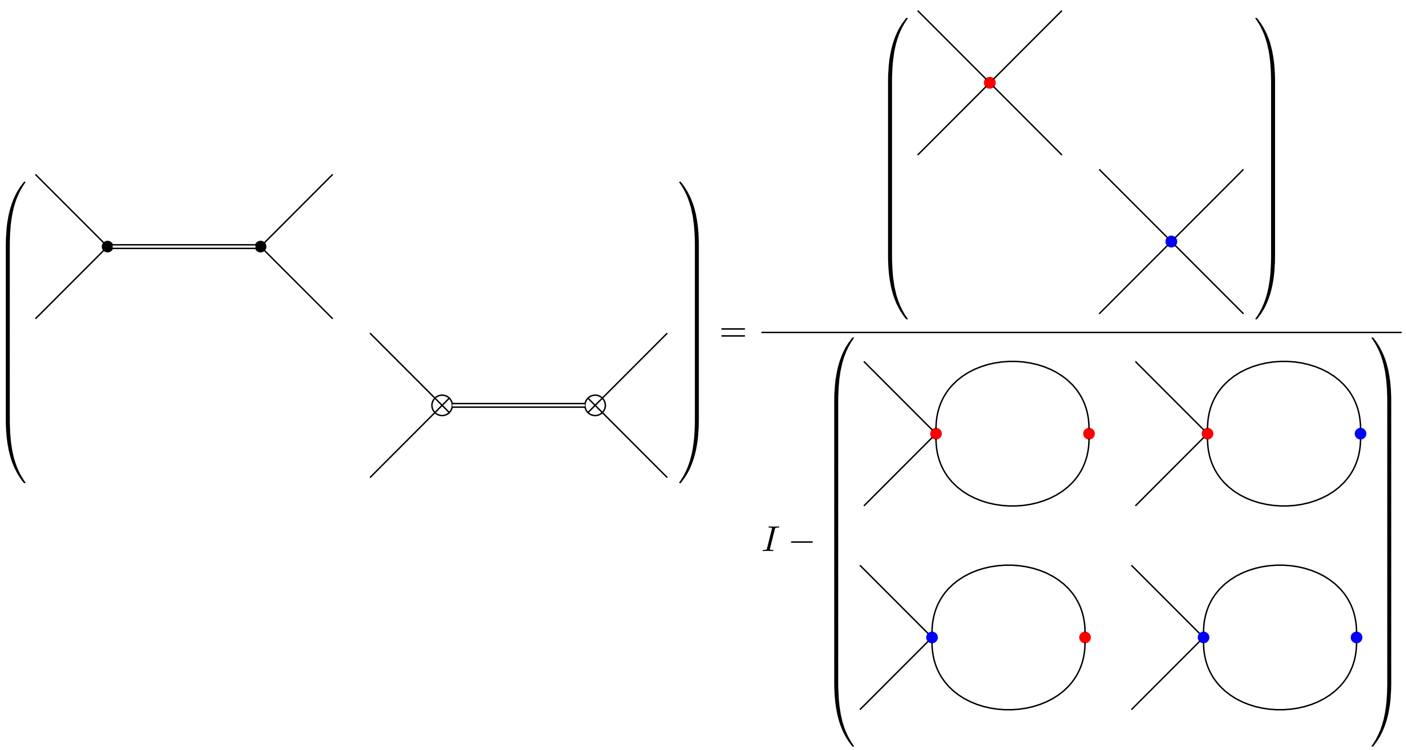
$$I_2 = N_c \sum_{q_f} \frac{|q_f|B}{8\pi^3} \sum_{\pm} \sum_{n=1}^{\infty} \int d^2k_{\shortparallel} \frac{2M}{(|k_{\shortparallel}| \pm \kappa)^2 - M_n^2};$$

$$I_3 = N_c \sum_{q_f} \frac{|q_f|B}{8\pi^3} \sum_{\pm} \sum_{n=1}^{\infty} \int d^2k_{\shortparallel} \frac{2(\kappa \pm |k_{\shortparallel}|)}{(|k_{\shortparallel}| \pm \kappa)^2 - M_n^2}.$$

# Random phase approximation

Quantum fluctuations: $\frac{\mathrm{tr}\ln G^{-1}}{\delta\phi} = \mathrm{tr}\sum_n \frac{(-1)^n}{n!}G^n\phi^n$ to $\phi^2 \sim \delta\pi\delta\pi, \delta\pi\delta\kappa, \delta\kappa\delta\kappa$, the geometry summation gives ($g = G_S = G_T$):

$$1 - 2g\Pi_{ps}\left(m_\pi^2\right) = 0.$$



# Revisited dimension reduction in the meson sector

Generalized polarization of the neutral pion

$$\frac{1}{i}\Pi^{AB} = -N_c \sum_{q_f} \mathrm{tr}\left[ iG(p)\, i\gamma_5 \Gamma^A\, iG(q)\, i\gamma_5 \Gamma^B \right].$$

where $\Gamma^{(A,B)} = \left(I_4, \sigma^{12}\right)$, respectively.

$$\begin{aligned}
\frac{1}{i}\Pi^{SS} &\sim \int d^2 k_{\shortparallel} \frac{1}{|p_{\shortparallel}|(|p_{\shortparallel}| - M + \kappa)} + \frac{1}{|q_{\shortparallel}|(|q_{\shortparallel}| - M + \kappa)} + \ldots \\
&\sim \int \frac{d^2 k_{\shortparallel}}{k_{\shortparallel}^2 - (M-\kappa)^2} + \ldots \\
&= \frac{I_1}{M-\kappa} + \frac{I_2}{M} - m_\pi^2 \langle J \rangle_0 - m_\pi^2 \langle K \rangle_n
\end{aligned}$$

# Detail results of polarization amplitude

In sum, the loop amplitude changes to

$$\begin{pmatrix} 1-2g\Pi^{SS} & -2g\Pi^{ST} \\ -2g\Pi^{TS} & 1-2g\Pi^{TT} \end{pmatrix} = \mathbf{A} + m_\pi^2 \mathbf{B}$$

where

$$\mathbf{A} = 1 - 2g\Pi_{ps}\big|_{m_\pi^2=0} = \begin{pmatrix} \eta & 0 \\ 0 & 2-\eta \end{pmatrix} - \frac{2\,igI_1}{M}\frac{1}{1-\xi}\begin{pmatrix} \xi & 1 \\ 1 & 2-\xi \end{pmatrix},$$

$$\mathbf{B} = 2\,ig\begin{pmatrix} \langle J\rangle_0 + \langle K\rangle_n & \langle J\rangle_0 + \langle K_2\rangle_n \\ \langle J\rangle_0 + \langle K_2\rangle_n & \langle J\rangle_0 + \langle K_1\rangle_n \end{pmatrix},$$

Note: $G_S = G_T = g$, $\eta = \frac{m}{M}$ and $\xi = \frac{\kappa}{M}$. $\mathbf{B}$ is slowly varying function.

# Possible pion superfluid

Rewrite

$$\mathbf{B} = 2\,ig\tilde{J}\begin{pmatrix} 1 & 1-\alpha \\ 1-\alpha & 1-\beta \end{pmatrix},$$

Applying the approximation $\eta, \xi, \alpha, \beta \ll 1$, one gets the corresponding pole of meson as

$$\begin{aligned} m_{\tilde{\pi}}^2 &= \frac{m}{-2\,igM\tilde{J}} + \frac{m+\kappa+igI_1}{M}\frac{I_1}{M\tilde{J}} + \mathcal{O}(\alpha^1) \\ m_{\tilde{\pi}}^2 &= \frac{1}{-ig\,(2\alpha-\beta)\,\tilde{J}} + \mathcal{O}(\alpha^0) \end{aligned}$$

negative $\frac{I_1}{M\tilde{J}}$ and $\kappa_{\rm cr} \simeq \frac{mM}{2\,igI_1} - igI_1 - m$

# Summary

- The infrared dynamics has a strong hierarchy of meson ($\sim \ln m^2$) and quark ($\sim m \ln m^2$) sectors.

- Competition between superfluid and superconductor if vector order parameter develops under strong magnetic field.

- A possbile state of compact star rather than phase of Bose-Einstein condensate.

- Explore the new phases for QCD matter in $(1+1)$-D where the fluctuations of the condensate come from two Goldstone Bosons: pions and phonons.

Thank You for Your Attention!