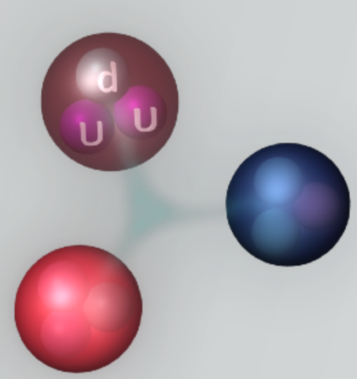
I.手征三体力与弱束缚原子核

II. 格点有效场论与有限温度核物质

马远卓

Institute of quantum matter,

South China Normal University



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粤港澳核物理论坛

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Shuang Zhang, Qi Yuan …

S. M. Wang (FUDAN)

Z. H. Sun (Oak Ridge)

B. S. Hu (TRIUMF)

Jianguo Li (IMP)

L. Coraggio (INFN)

A. Gargano (INFN)

N. Itaco (INFN)

N. Michel (IMP)

Nuclear Lattice EFT Collaboration

Dean Lee, Ulf-G. Meißner, Timo A. Lähde, Evgeny Epelbaum, Ning Li, Bingnan Lv, Serdar Elhatisari…

PKU: *School of Physics, Peking University, China*

INFN: *Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Italy*

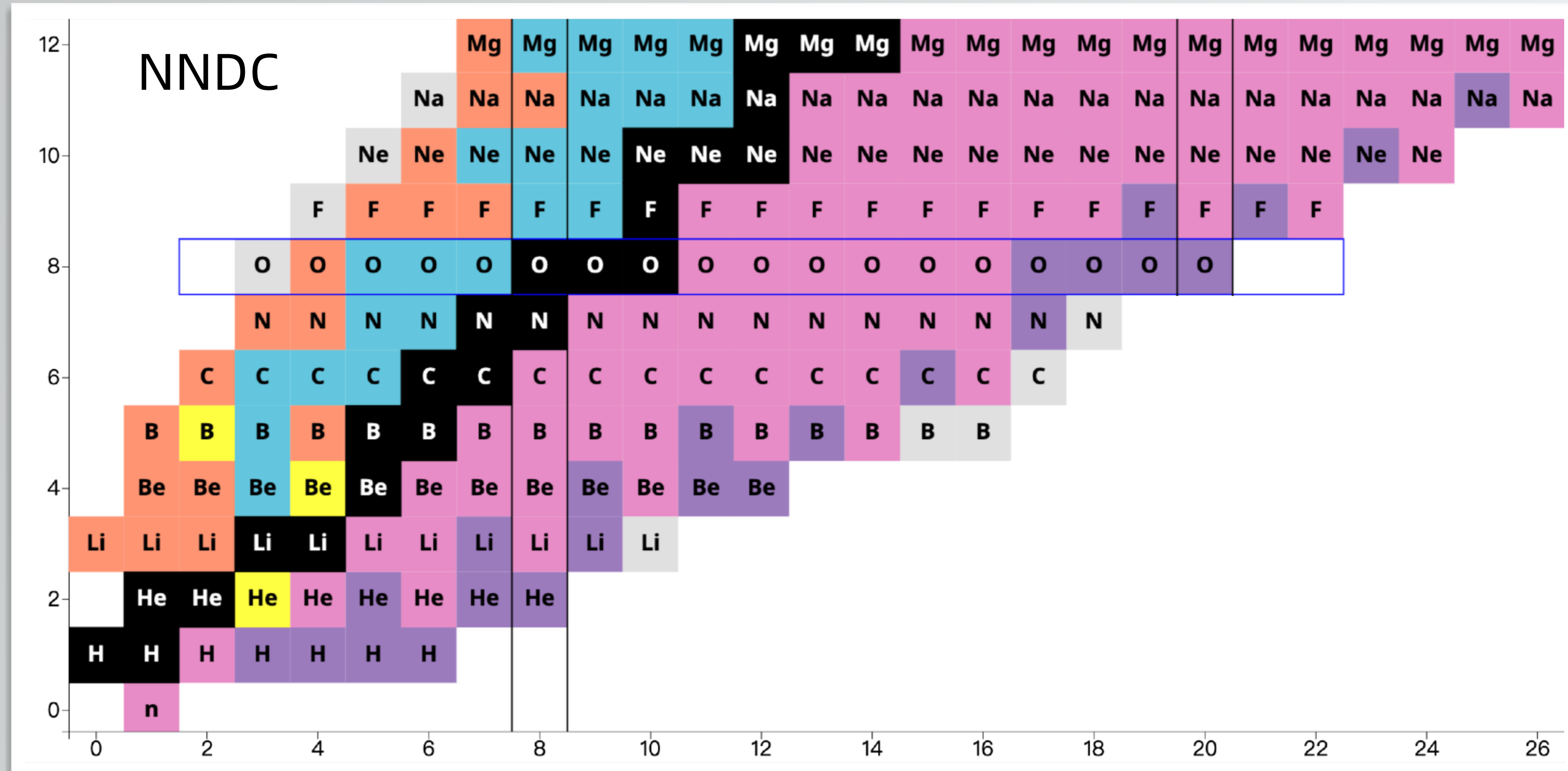
IMP: *Institute of Modern Physics, Chinese Academy of Sciences, China*

Oak Ridge: *Physics Division, Oak Ridge National Laboratory, USA*

TRIUMF: *TRIUMF, Canada*

Weakly bound nuclear systems

Experiment: rich phenomena & impressive progress



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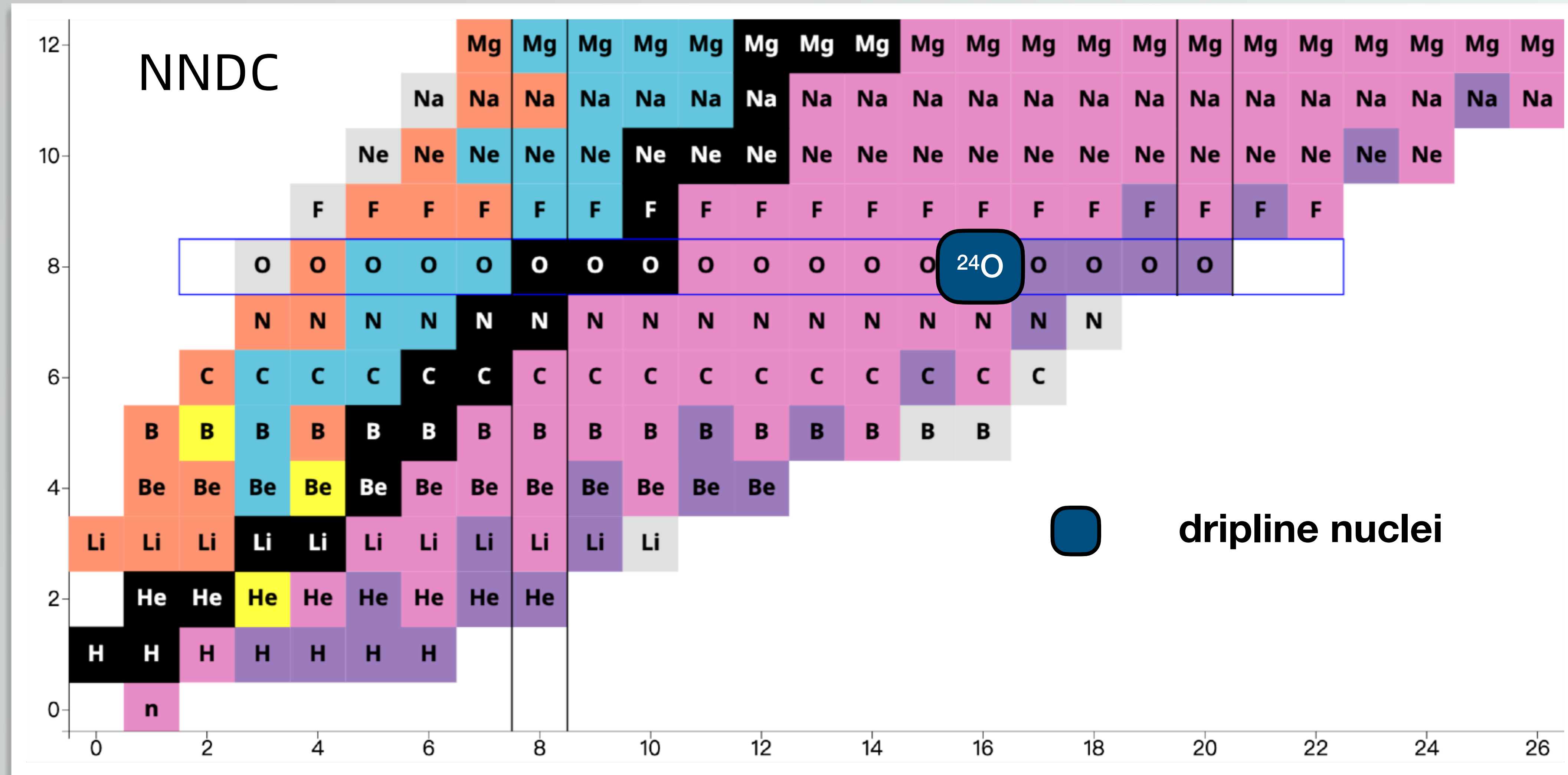

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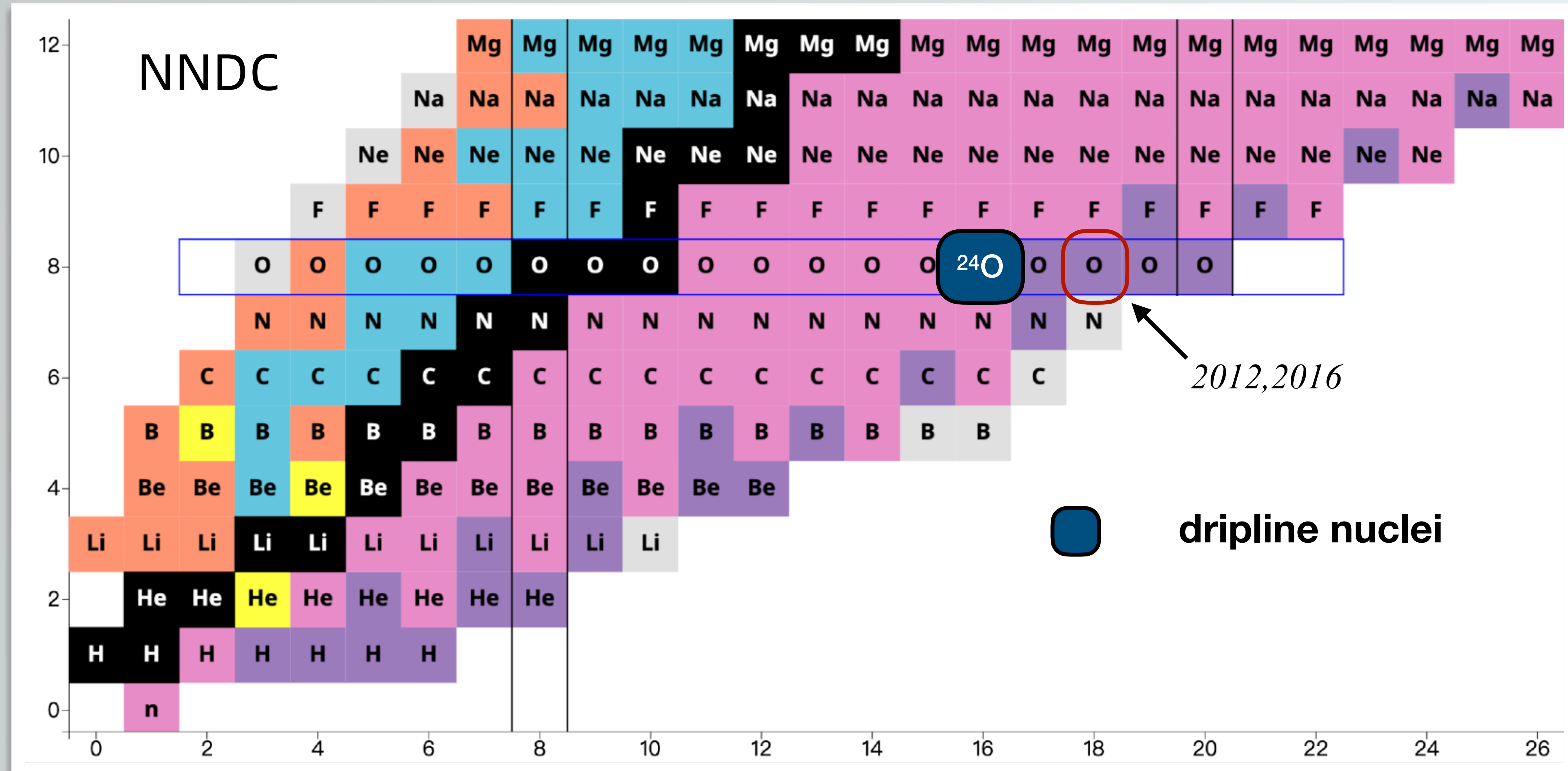

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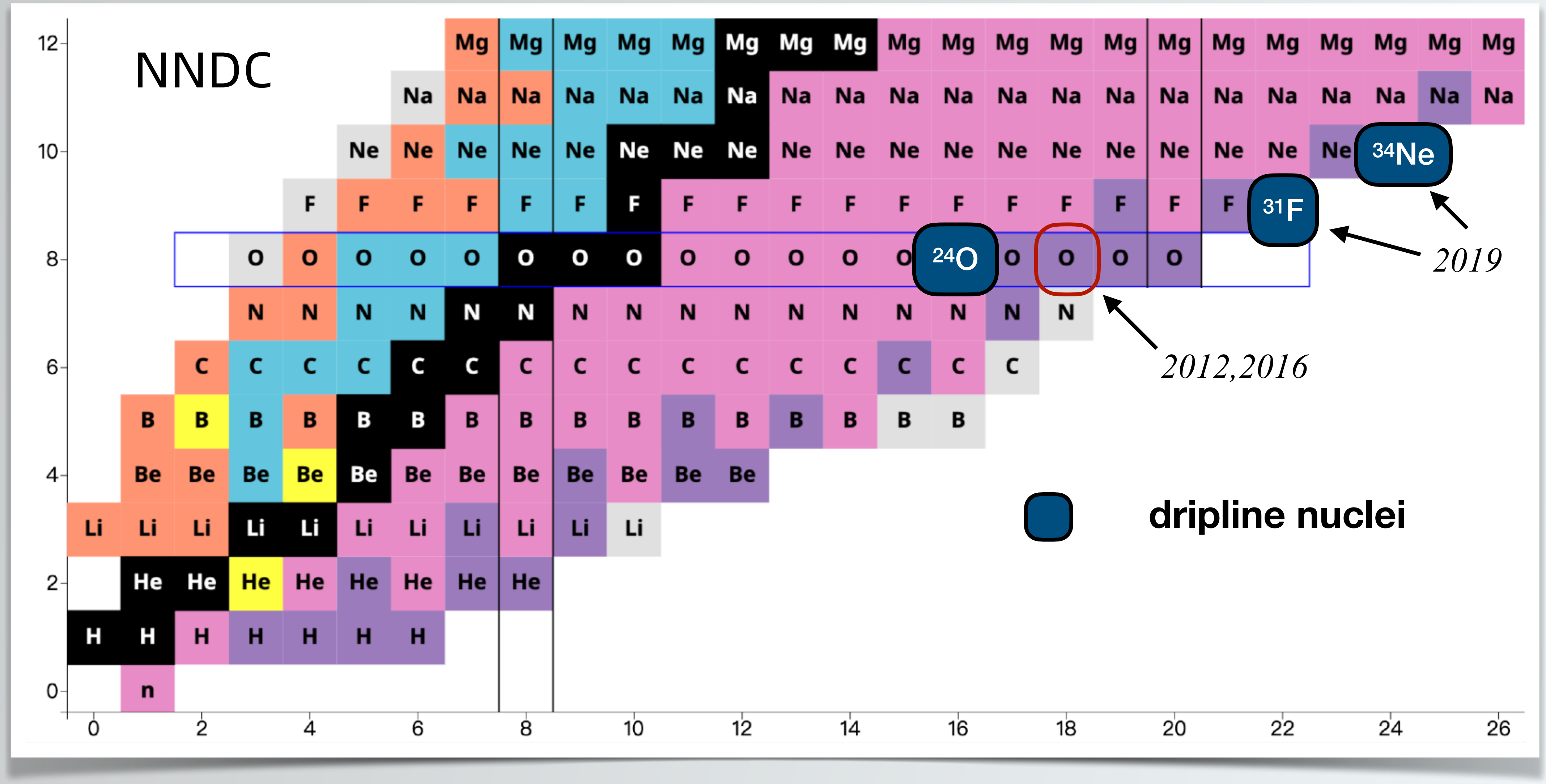

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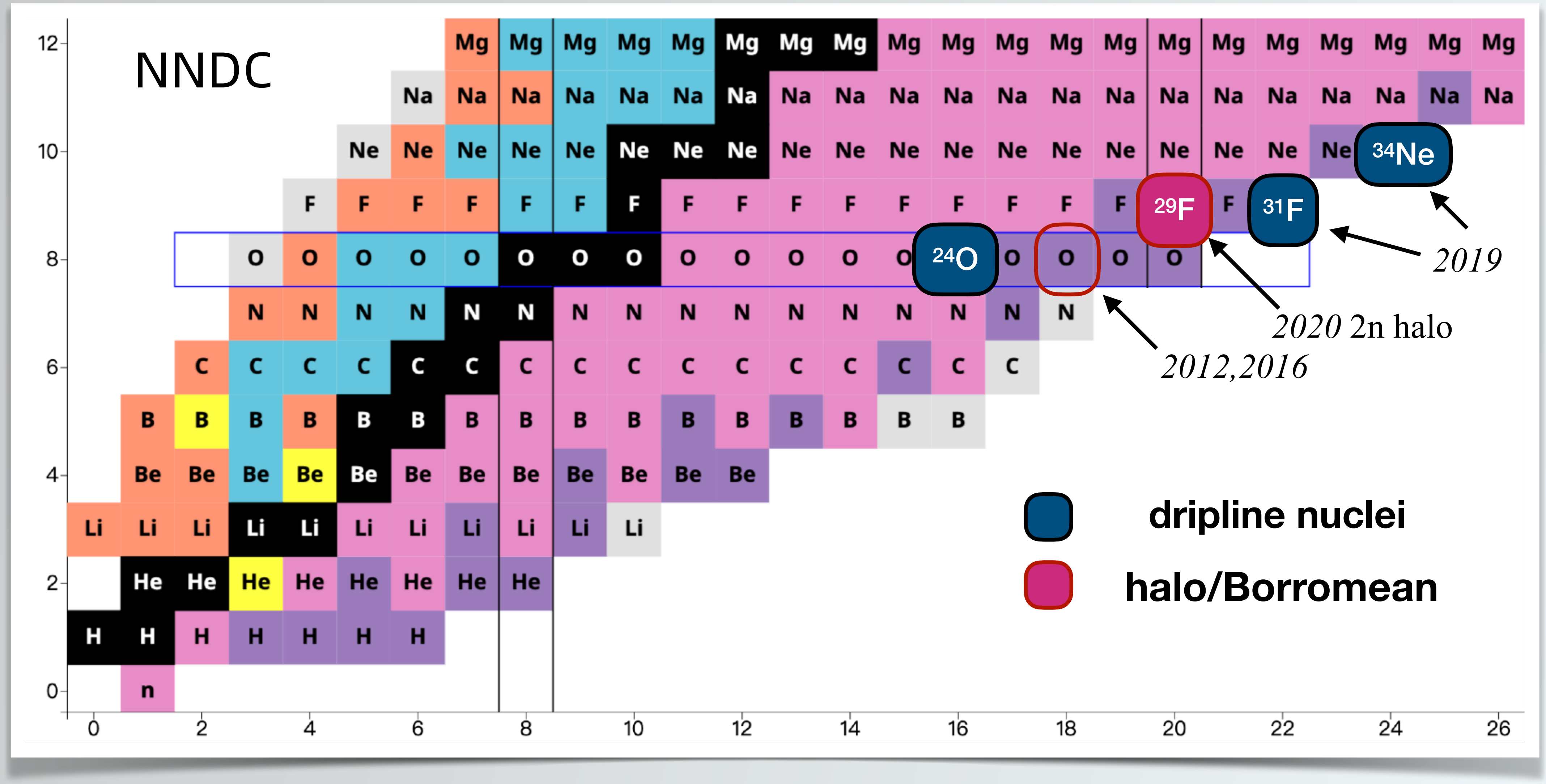

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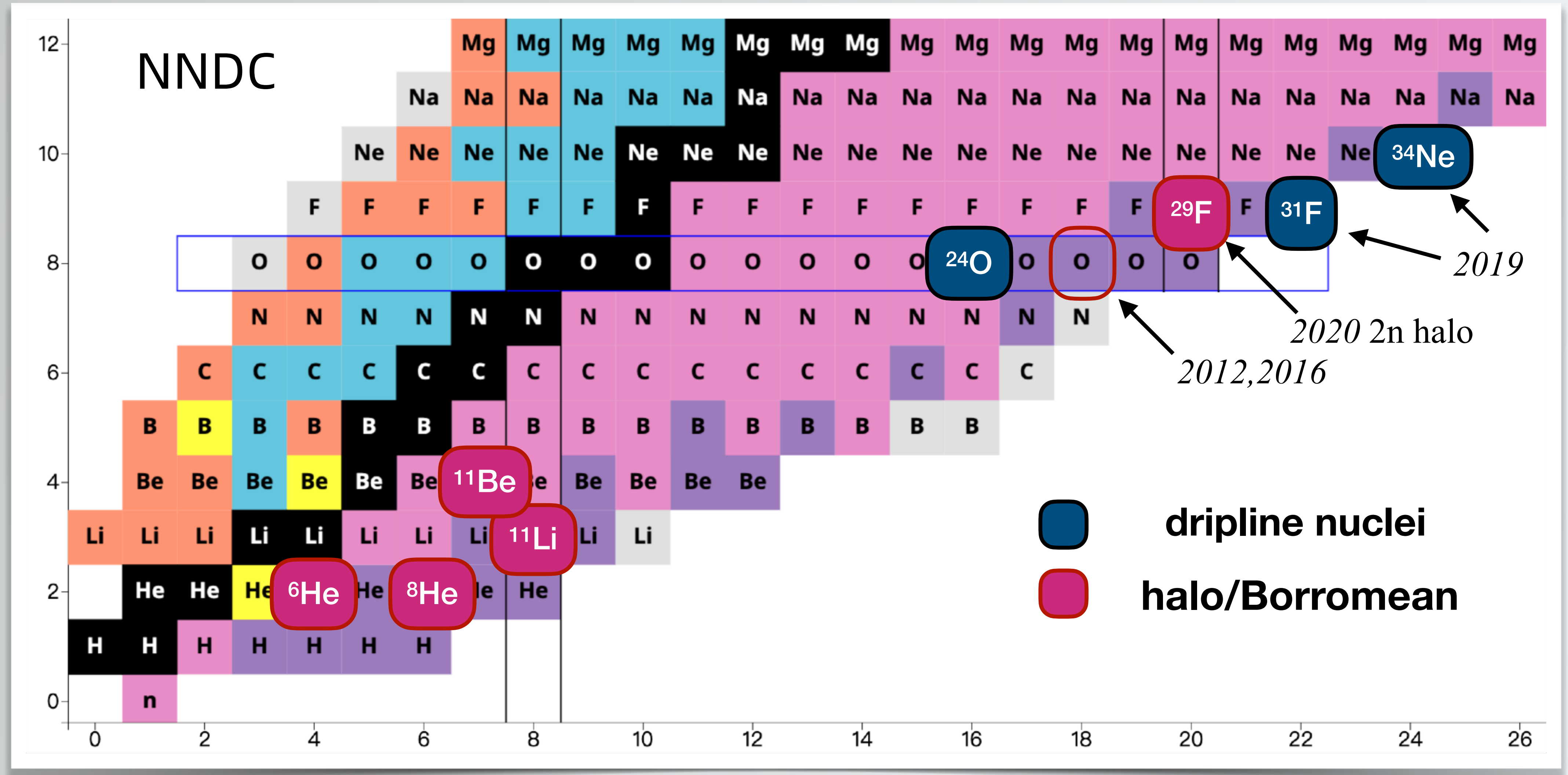

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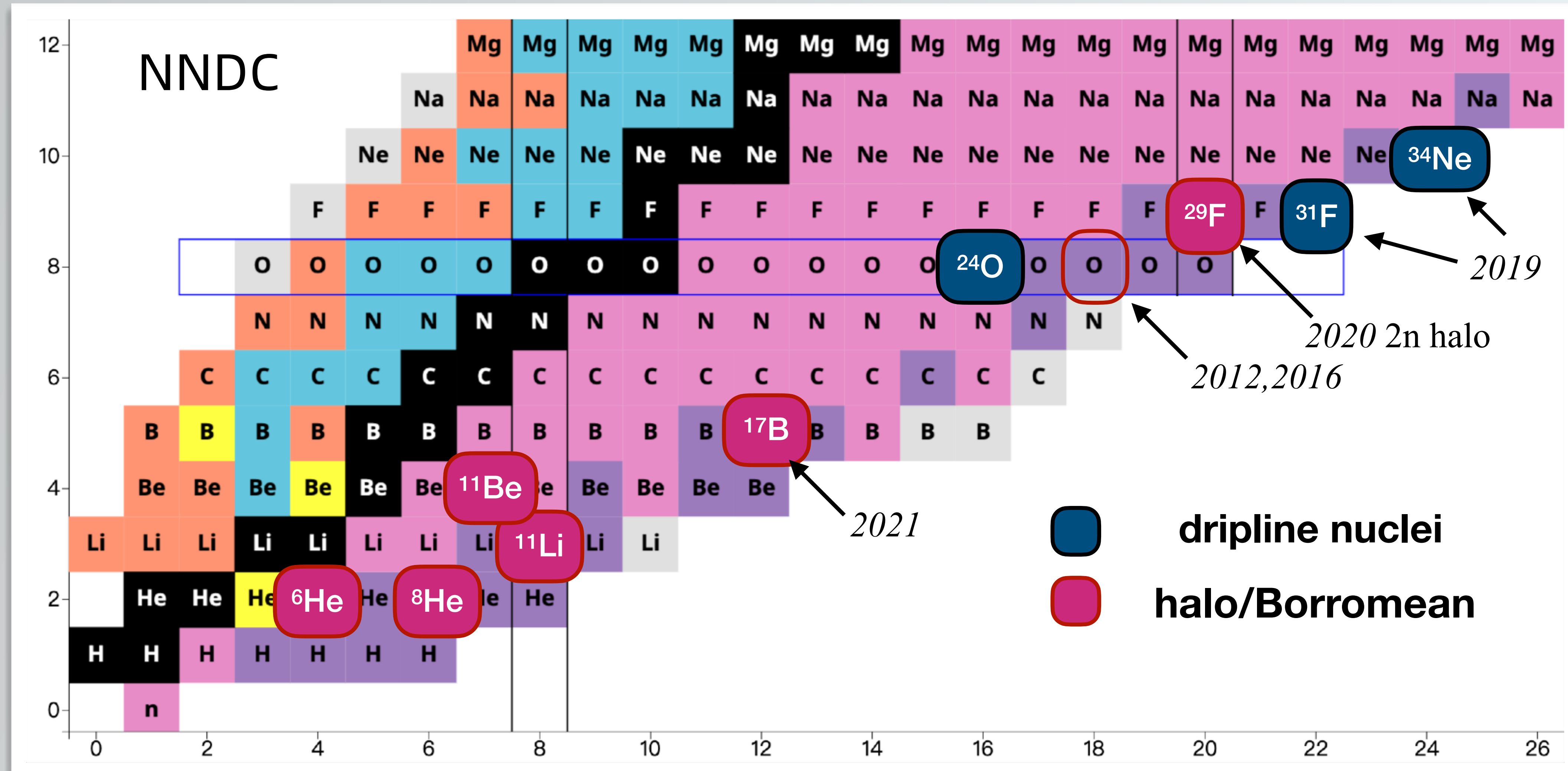

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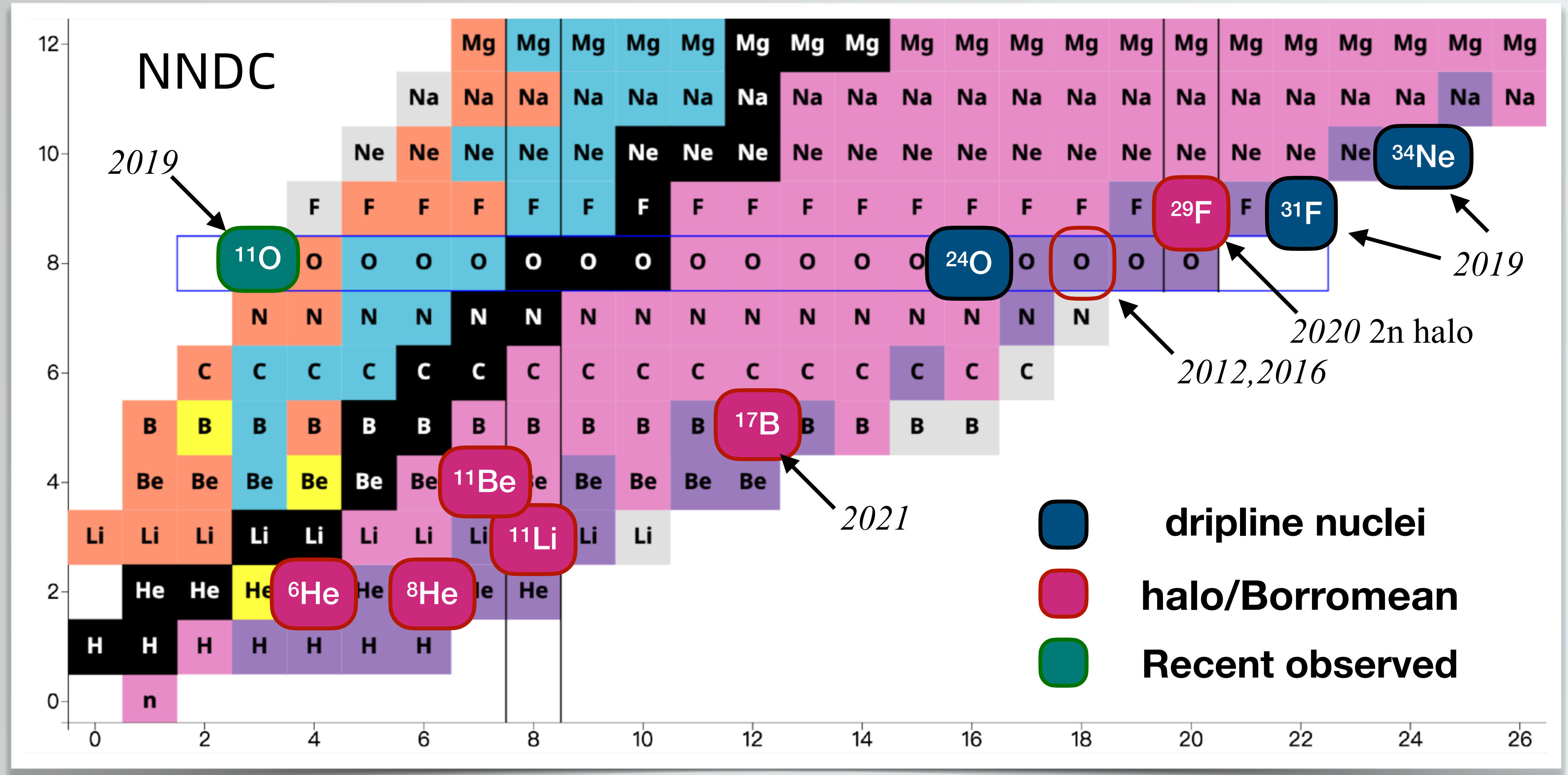

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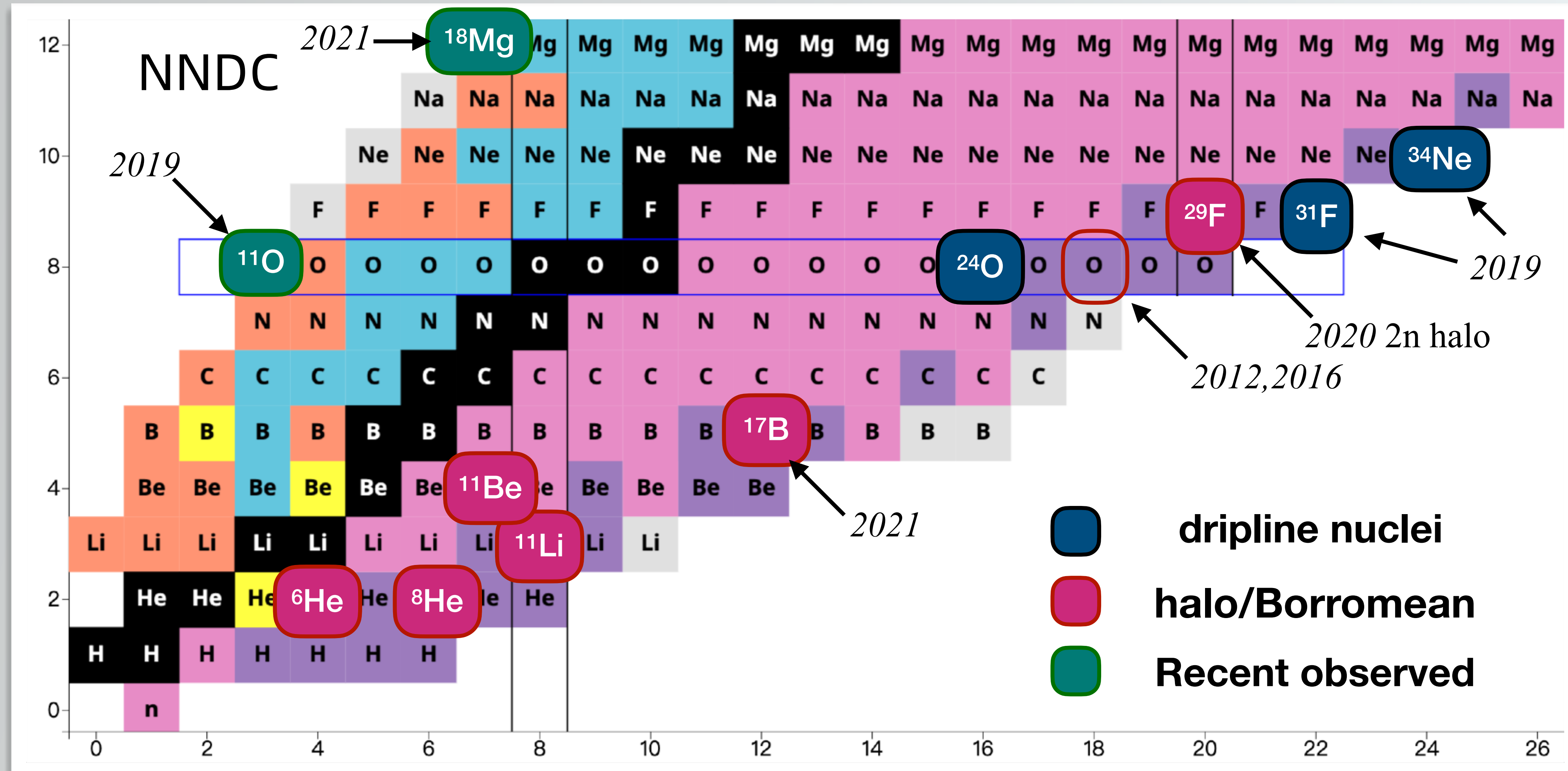

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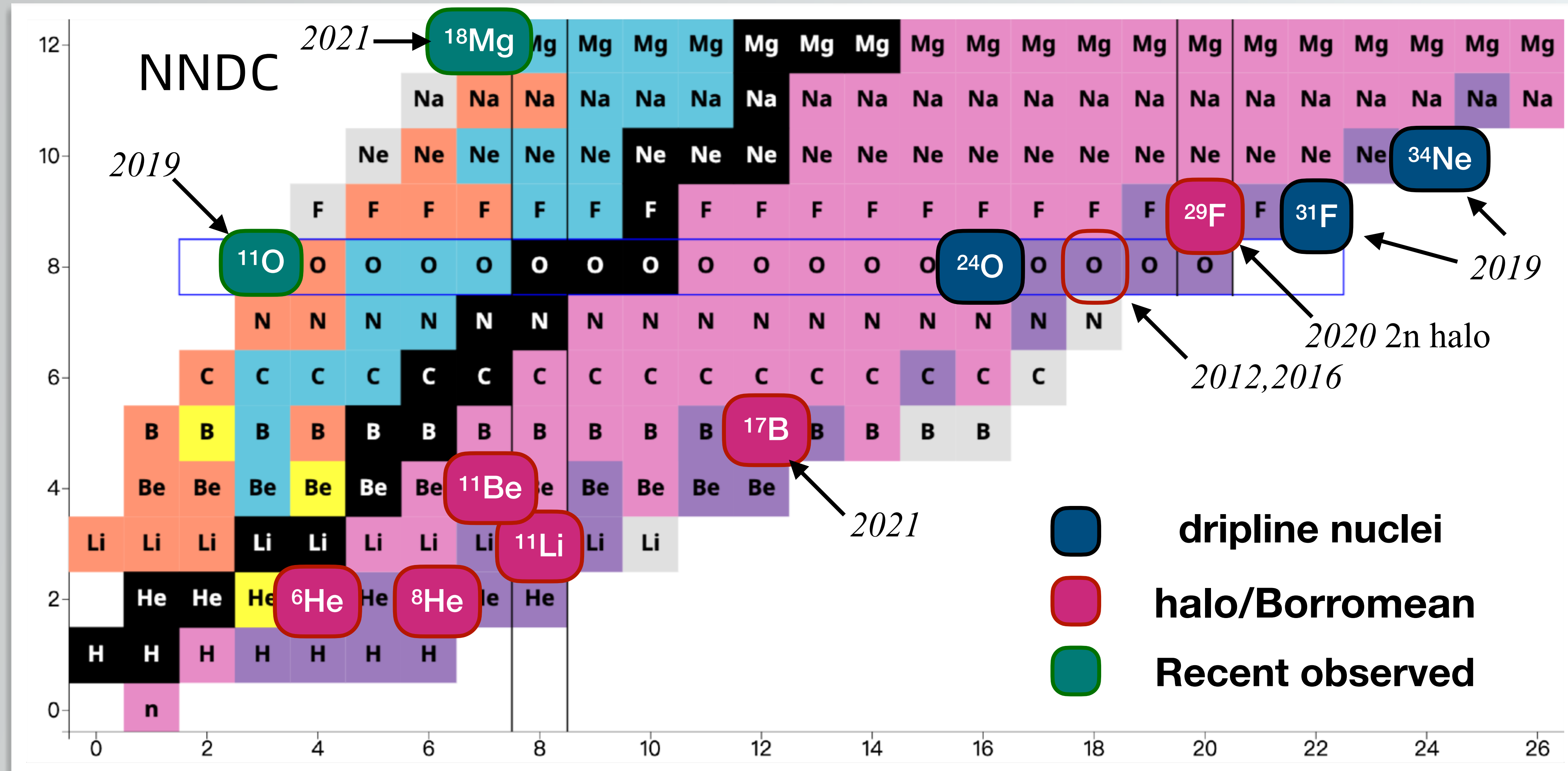

Theoretical Challenges: Nuclear Force & Quantum Many-body methods

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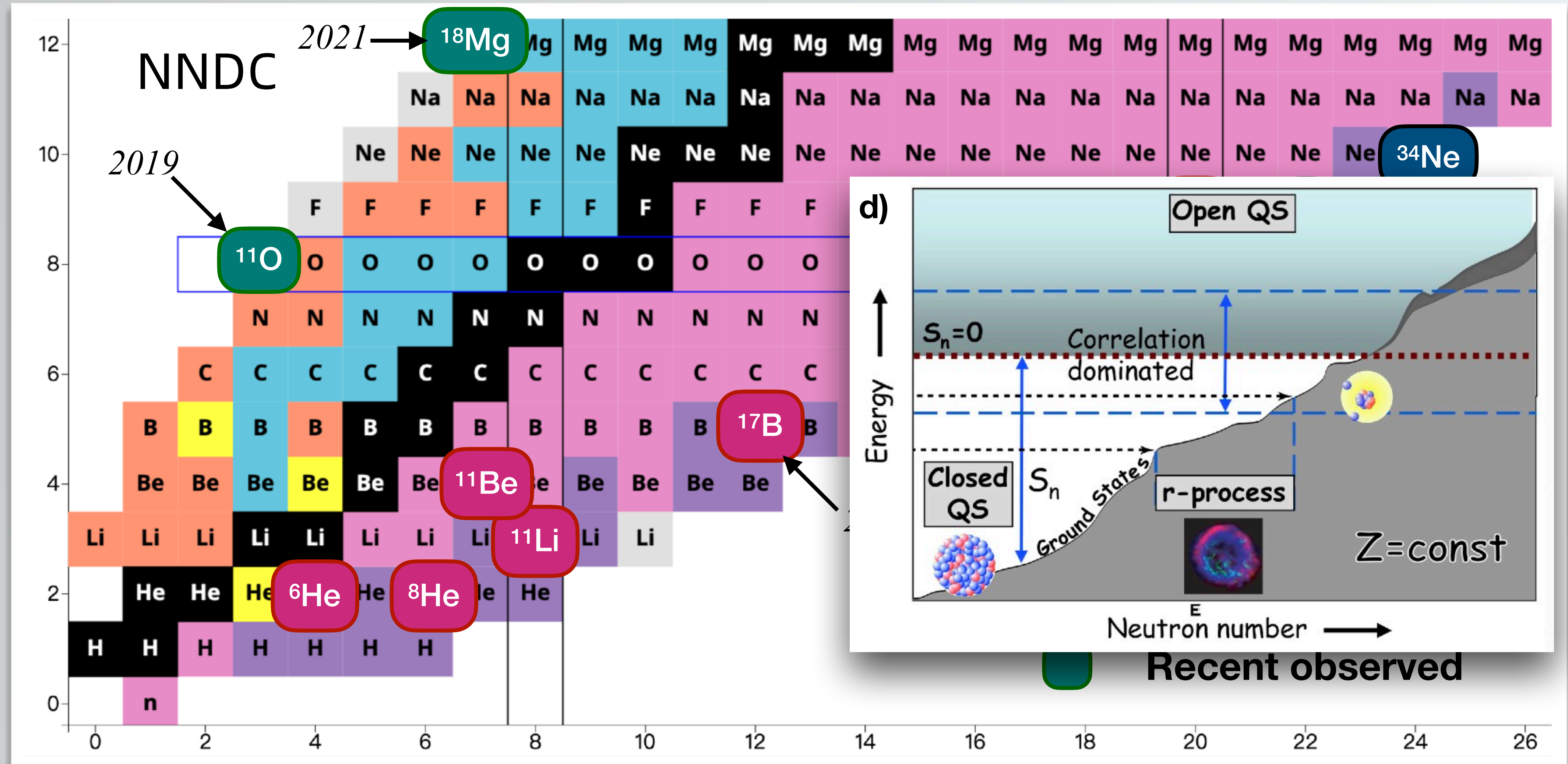

Theoretical Challenges: Nuclear Force & Quantum Many-body methods

d) N Michel, W Nazarewicz, et al., J. Phys. G **36** (2009) 013101

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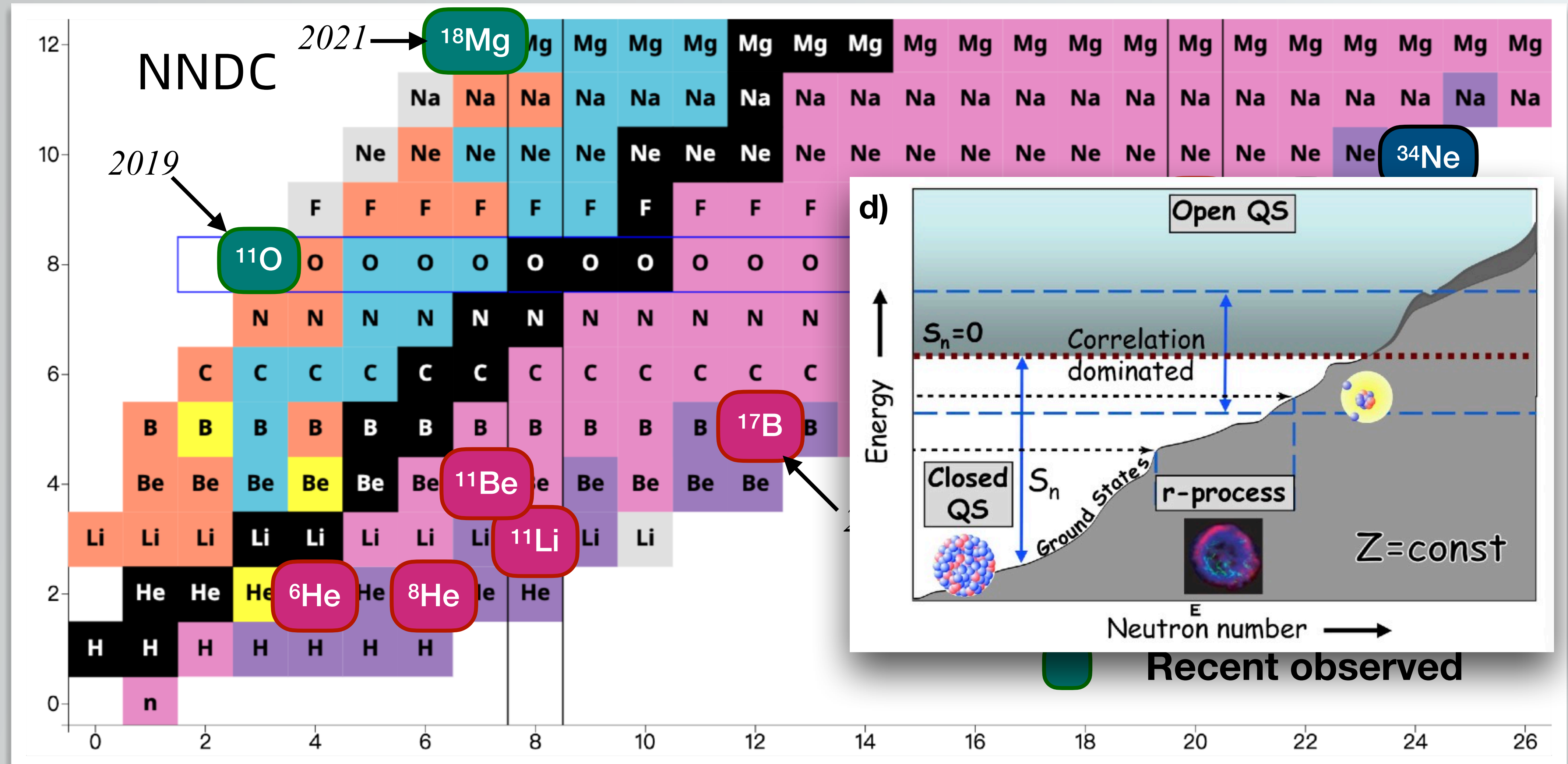

Theoretical Challenges: **Nuclear Force** **&** **Quantum Many-body methods**

Continuum

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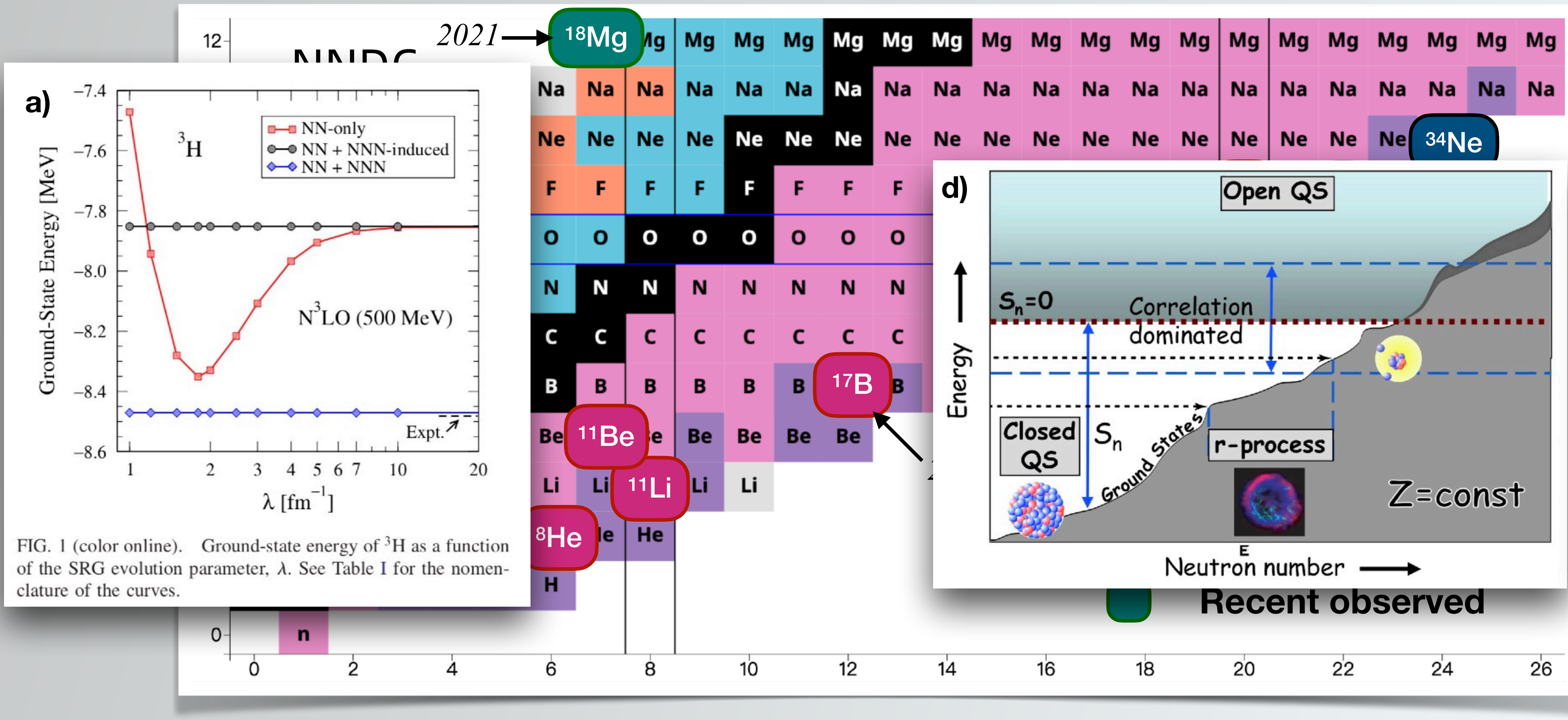

Theoretical Challenges: Nuclear Force & Quantum Many-body methods

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a) E. D. Jurgenson, P. Navrátil *et al.*, PRL **103**, 082501 (2009)

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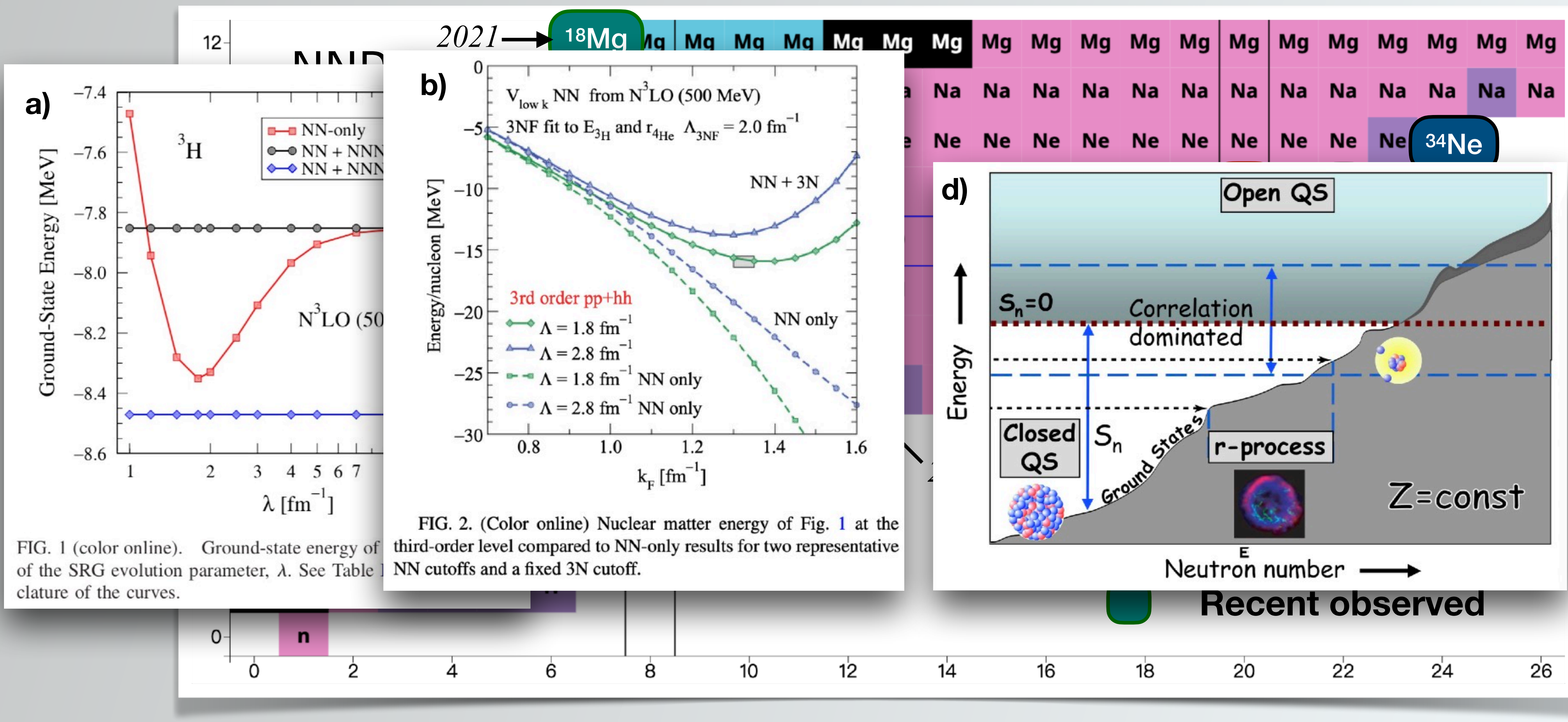

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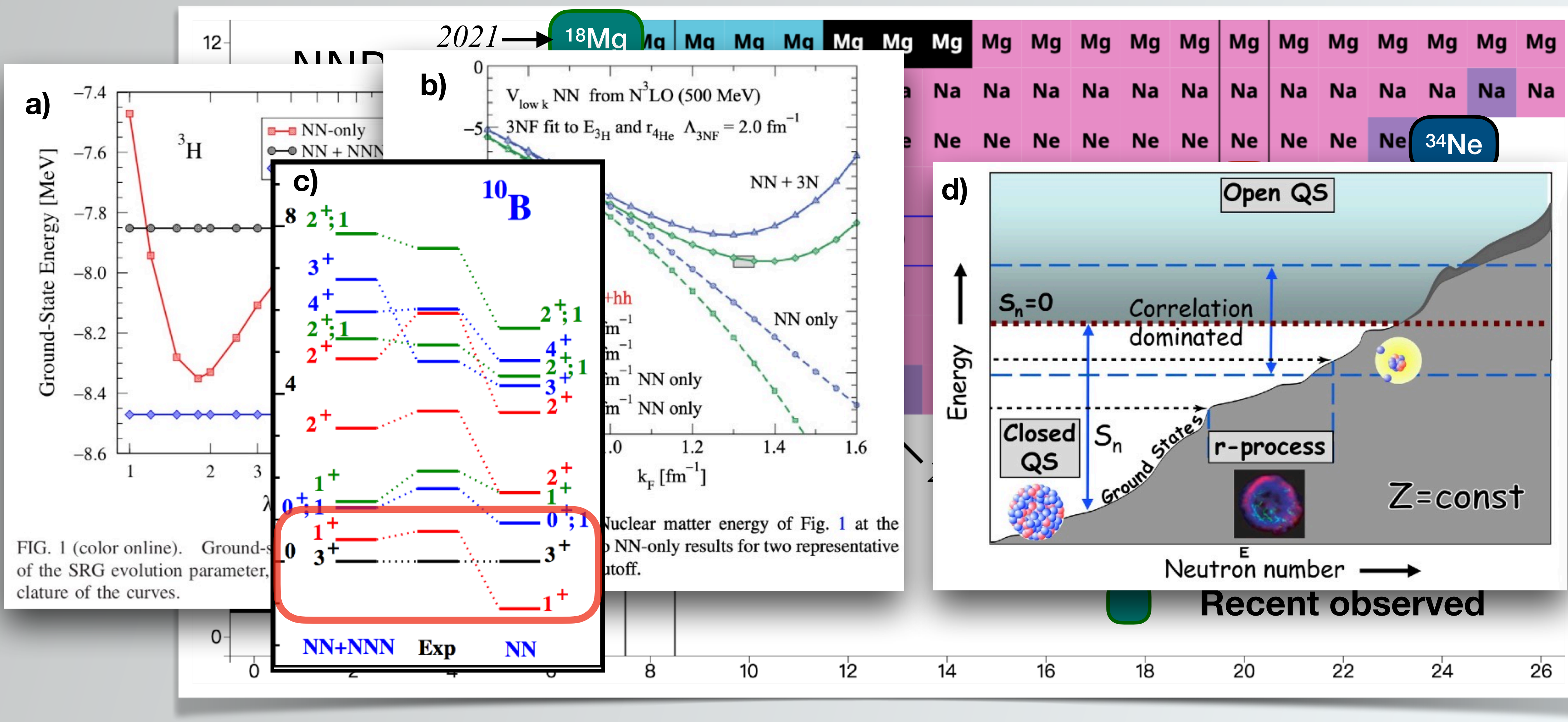

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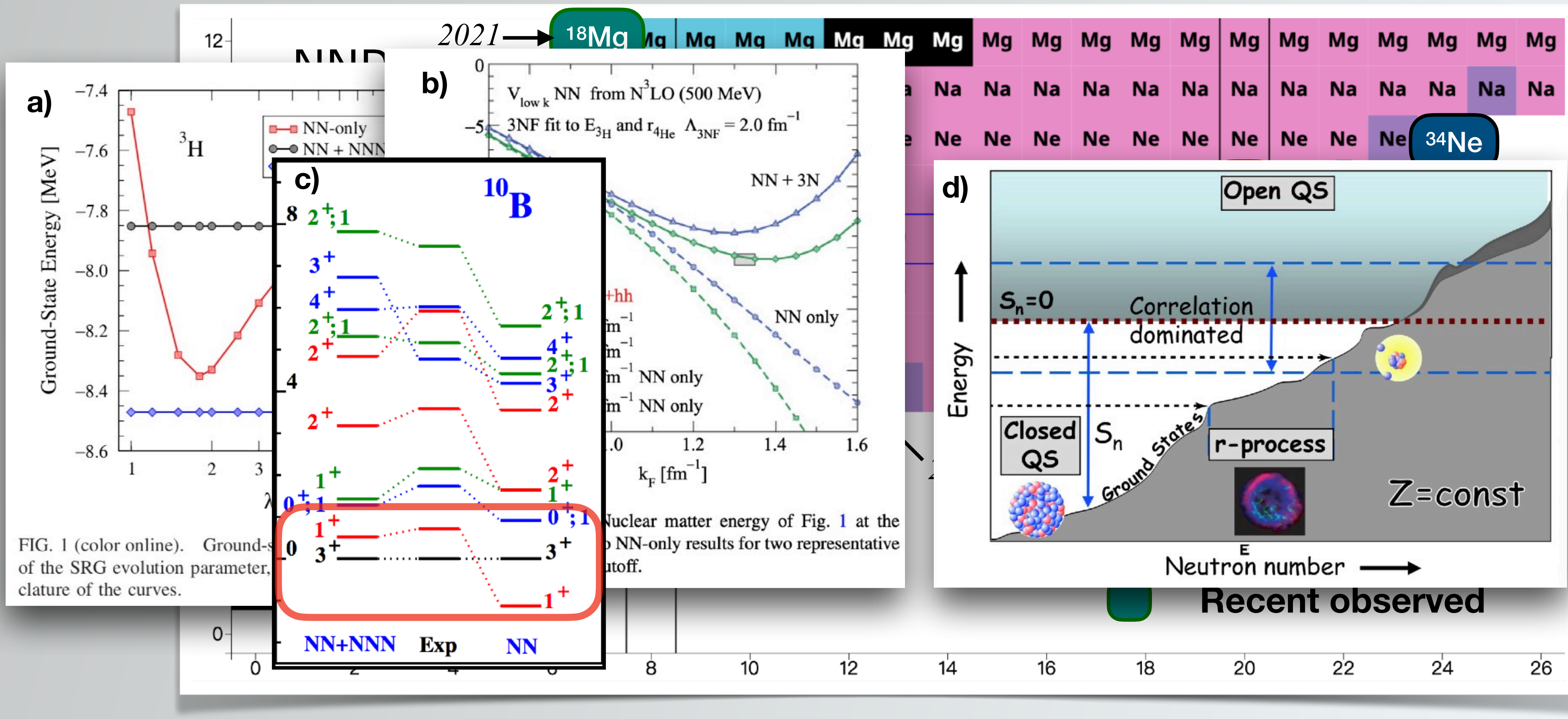

Theoretical Challenges: Nuclear Force & Quantum Many-body methods

3NF

Continuum

a) E. D. Jurgenson, P. Navrátil *et al.*, PRL **103**, 082501 (2009)

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Outline

Chiral 3NF

Chiral 3NF: origin, derivation, calculation & benchmark, implement (RSM or MBPT)

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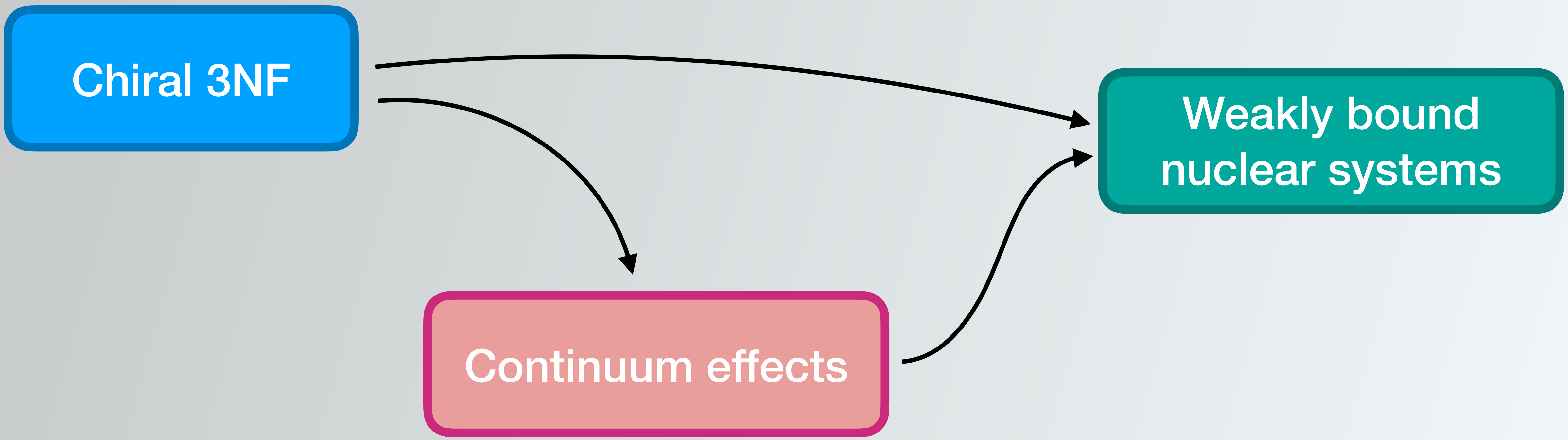

Chiral 3NF: origin, derivation, calculation & benchmark, implement (RSM or MBPT)

Continuum effect: Berggren basis, inclusion of 3NF, implement (GSM)

Weakly bound nuclear system:

1. neutron rich Oxygen isotopes
2. Borromean ^{17}Ne
3. Mirror symmetry breaking partners (张爽报告)

Nuclear Force is not the fundamental forces

Nuclear Force

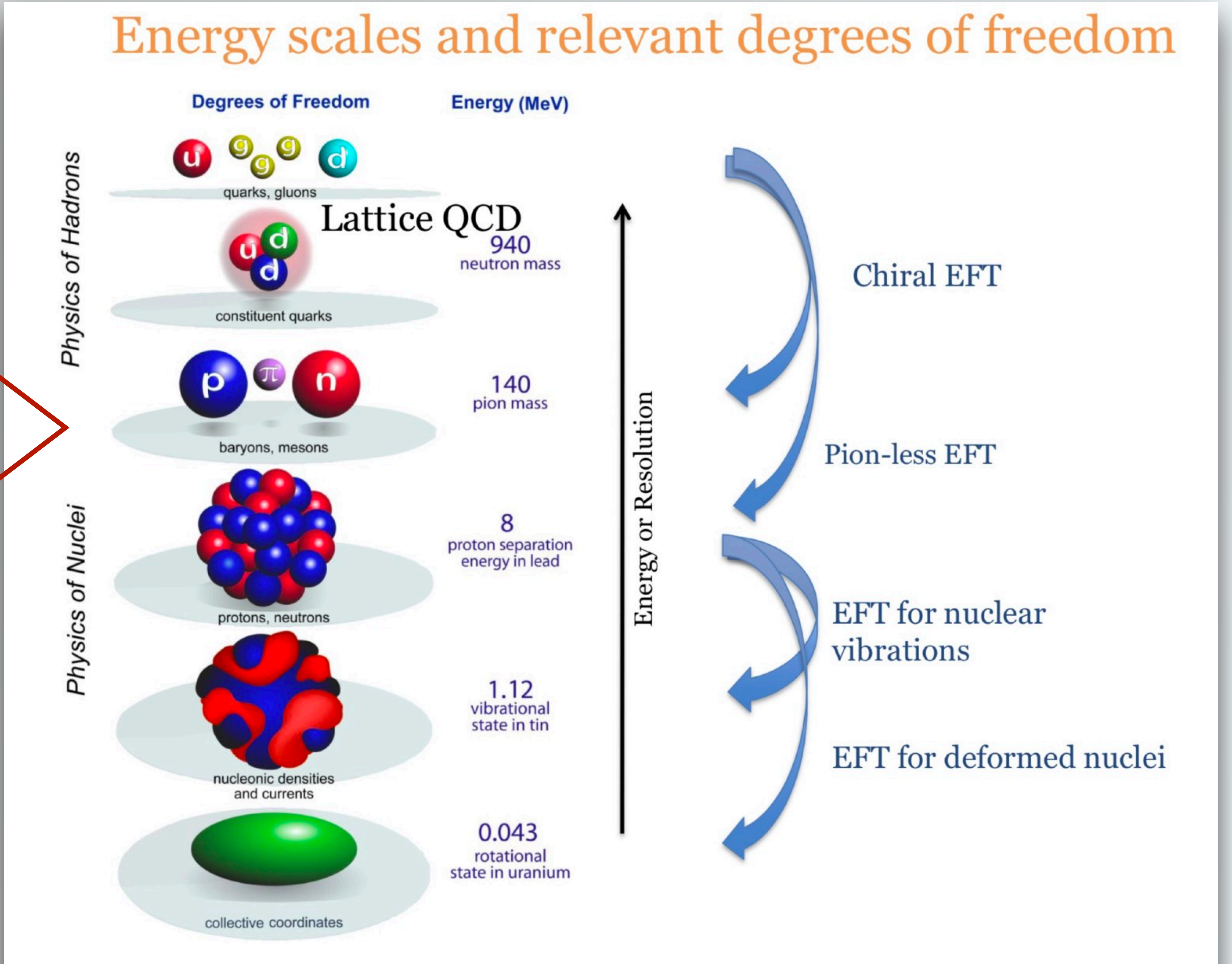

Fig.: Bertsch, Dean, Nazarewicz (2007)

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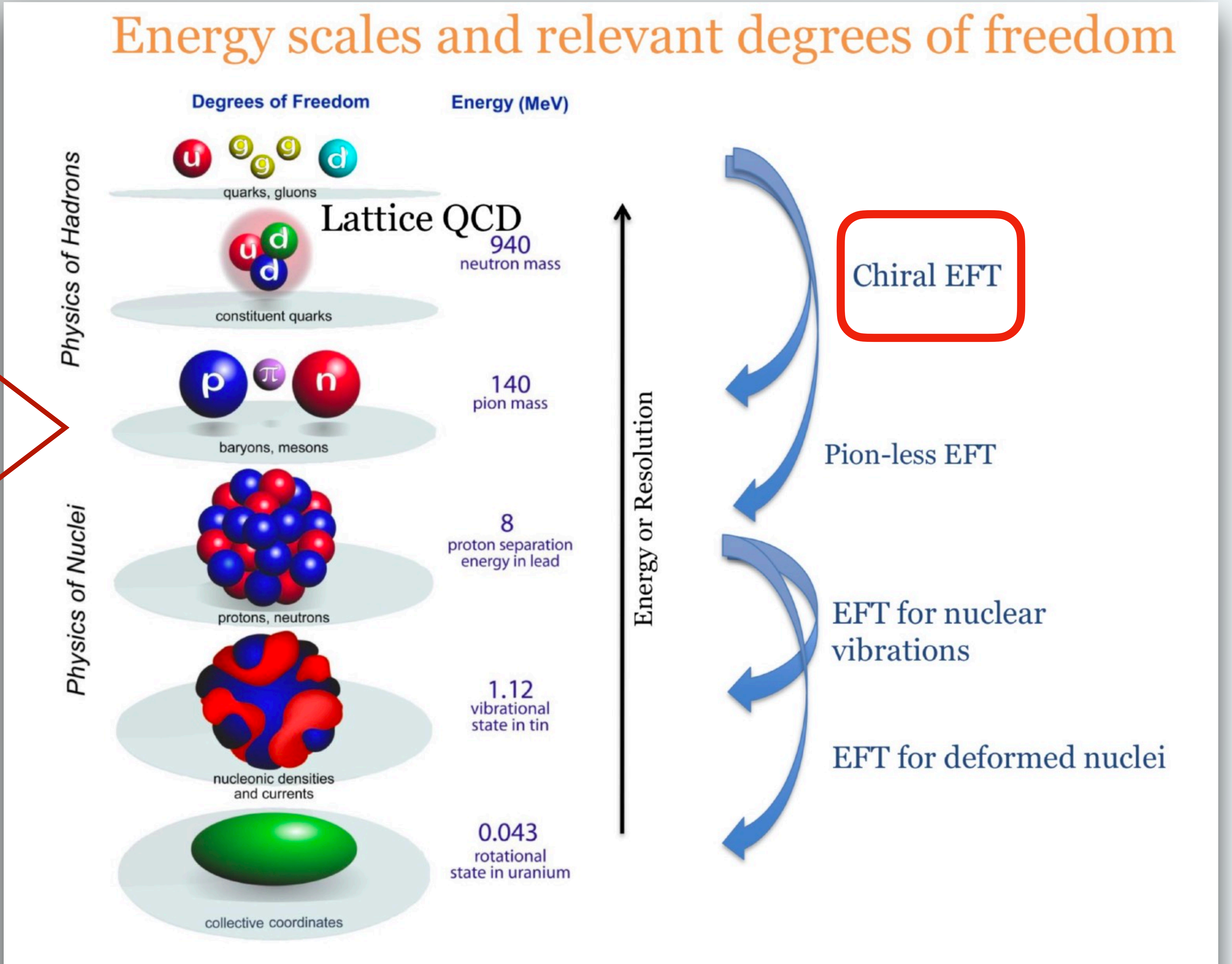


Fig.: Bertsch, Dean, Nazarewicz (2007)

3NF from Chiral EFT

QCD and **nuclear physics** can be linked by **Chiral EFT**

Effective Lagrangians (for nuclear forces)

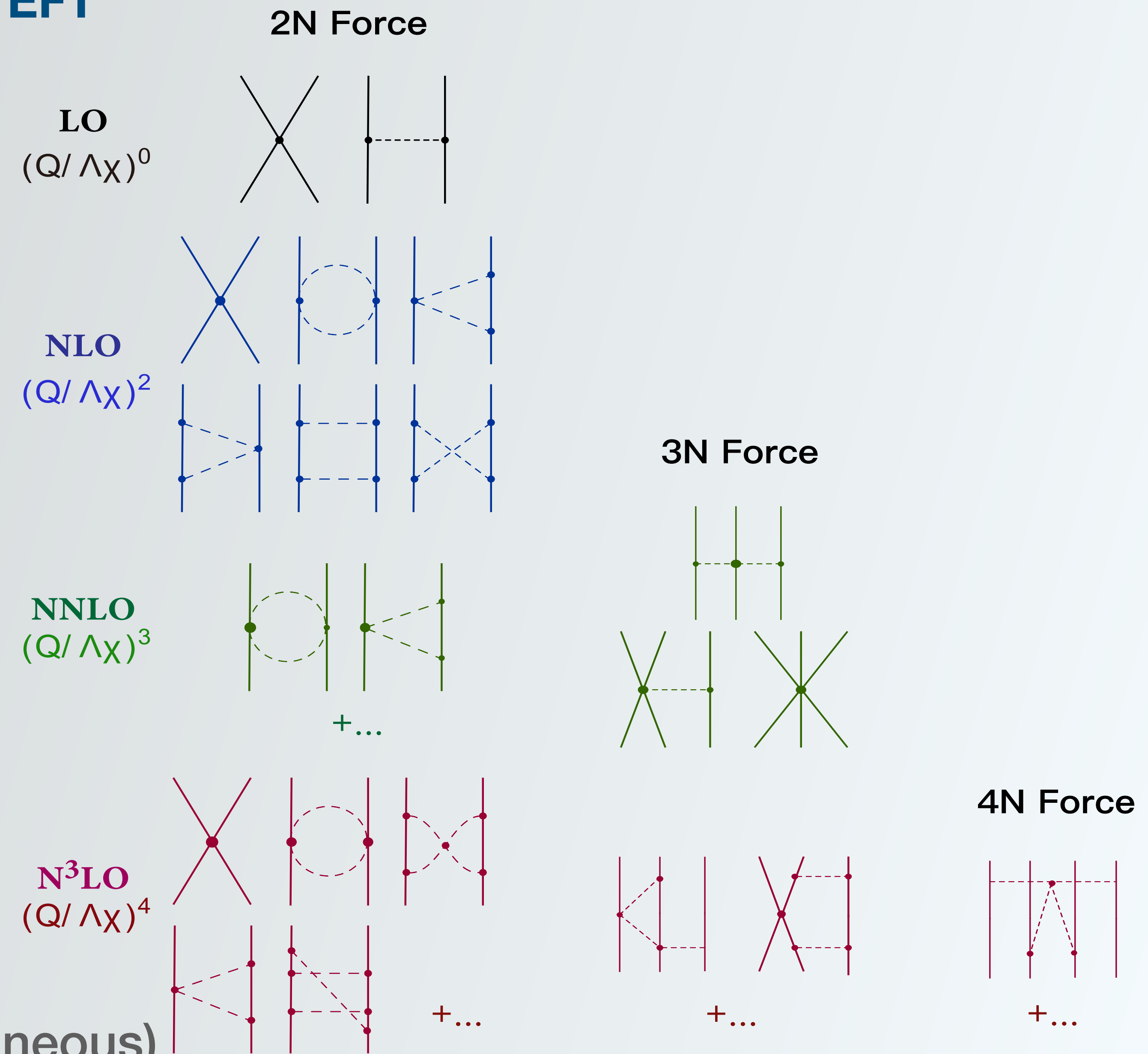
$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\pi\pi} + \mathcal{L}_{\pi N} + \mathcal{L}_{NN} + \cdots$$

Infinitely many terms

A scheme to make the theory **manageable** and **calculable**

Chiral perturbation theory (ChPT)

- Degree of freedom : nucleons and pions
- Chiral symmetry and breaking (explicit/spontaneous)
- Many body forces on equal footing



Weinberg; van Kolck; R. Machleidt; D. Entem *et al*.

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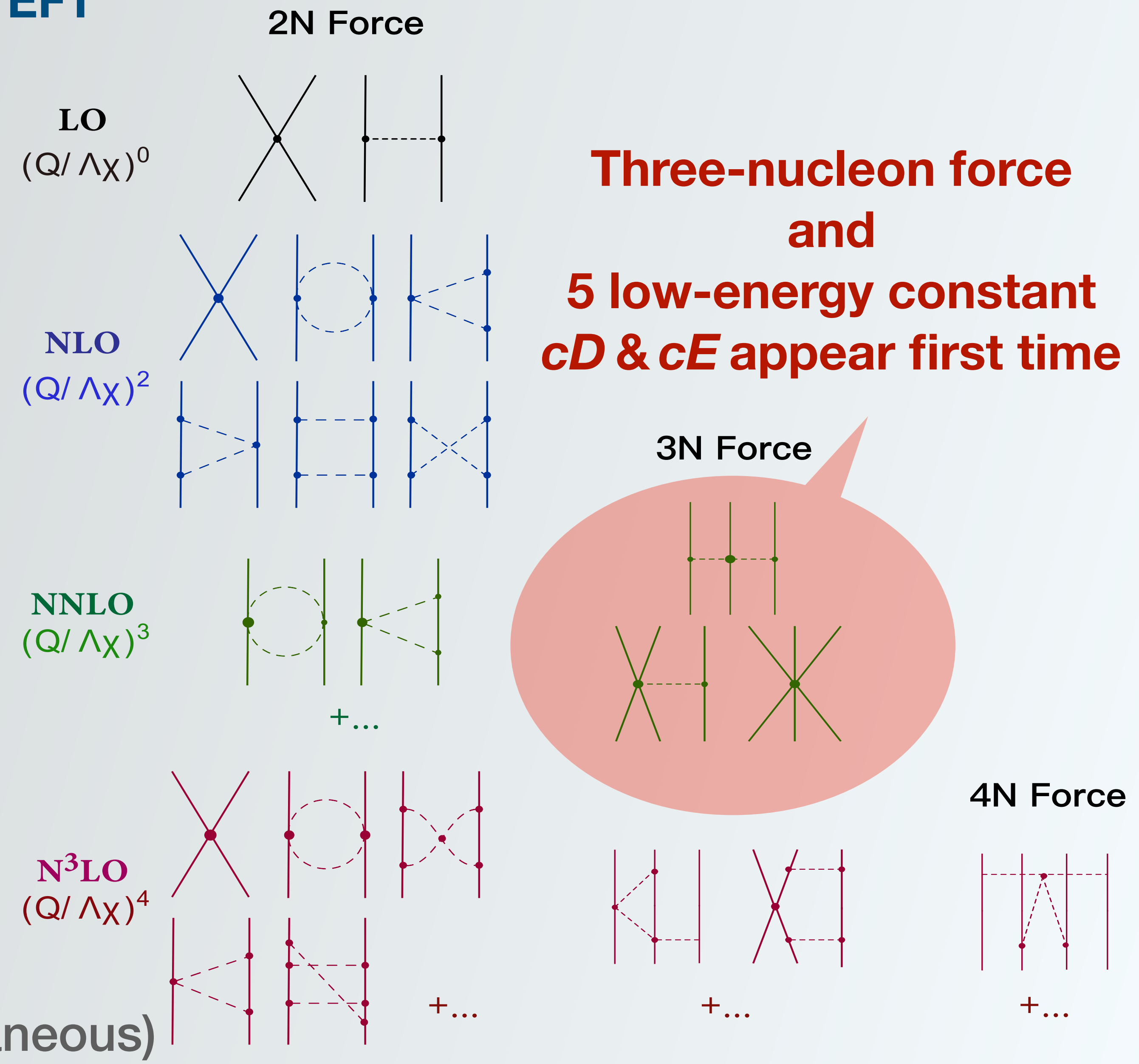
From N²LO, three-nucleon force (3NF) appears



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Purpose & Procedure

Purpose

- **Including 3NF based on EFT in nuclear many body calculations by means of the harmonic-oscillator basis.**
- **Investigating 3NF effects and the dependence of cut off (regulator), LECs, model space, etc.**

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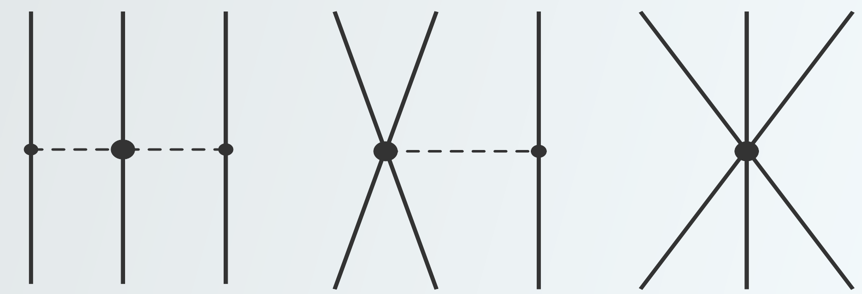
Procedure

$$ {}_{A}\langle [[\bullet\bullet]\bullet]_{JT} | V_{3N} | [[\bullet\bullet]\bullet]_{JT} \rangle_{A} $$



$V_{3N}^{2\pi-exch}$ $V_{3N}^{1\pi+contact}$ $V_{3N}^{contact}$

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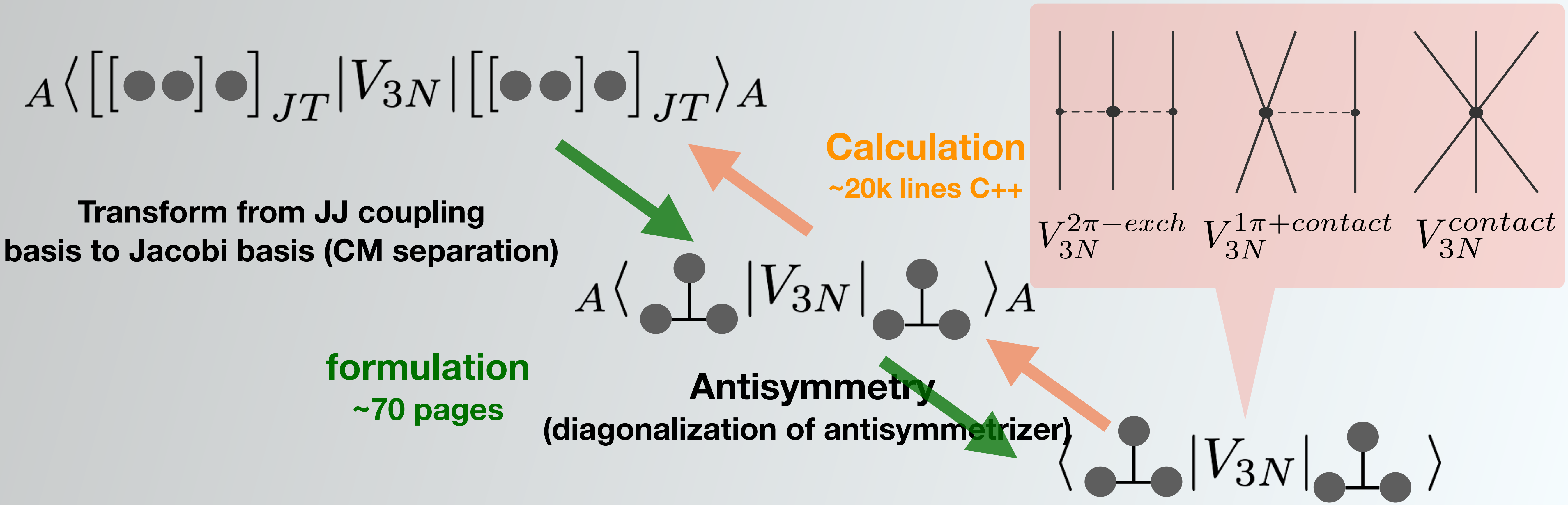


Generating 3NF matrix element

$$
{}_{as}\langle \tilde{a}'\tilde{b}'\tilde{c}'; J'_{ab} J T'_{ab} T | V_{3\mathrm{N}} | \tilde{a}\tilde{b}\tilde{c}; J_{ab} J T_{ab} T \rangle_{as} = 6 \sum_{\substack{N_{12}, N_3, \alpha \\ N'_{12}, N'_3, \alpha'}} \sum_{N_0 L_0} \sum_{i, i'} \delta_{T_{ab} T_{12}} \delta_{T'_{ab} T'_{12}}
$$

T coefficients: Jacobi -> (abc)JT state

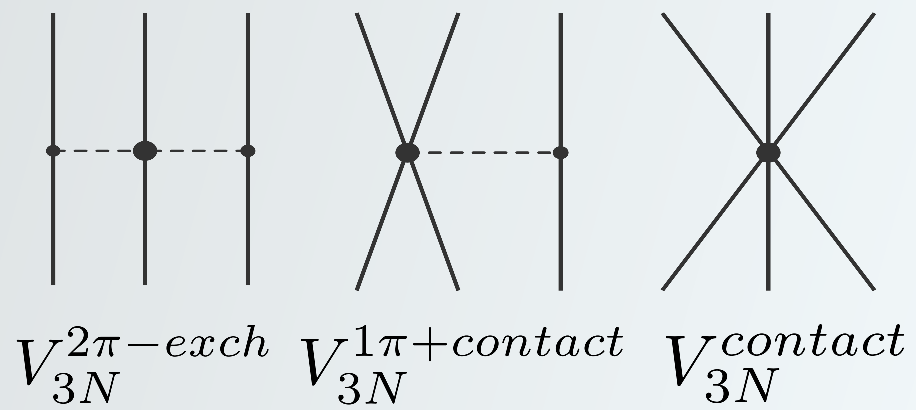
$$
\times\, T^{\tilde{a}'\tilde{b}'\tilde{c}' J'_{ab} J}_{N'_{12} N'_3 \alpha' N_0 L_0} T^{\tilde{a}\tilde{b}\tilde{c} J_{ab} J}_{N_{12} N_3 \alpha N_0 L_0}
$$

M coefficients: Antisymmetry

$$
\times\, M^{i'}_{N'_{12} N'_3 \alpha'} M^{i}_{N_{12} N_3 \alpha}
$$

3BME in Jacobi basis

$$
\times\, \langle N' i' J_{12,3} T, N_0 L_0; J | V_{3\mathrm{N}} | N i J_{12,3} T, N_0 L_0; J \rangle
$$



Generating 3NF matrix element

$$ {}_{as}\langle \tilde{a}'\tilde{b}'\tilde{c}'; J'_{ab}JT'_{ab}T | V_{3\mathrm{N}} | \tilde{a}\tilde{b}\tilde{c}; J_{ab}JT_{ab}T\rangle_{as} = 6 \sum_{\substack{N_{12},N_3,\alpha \\ N'_{12},N'_3,\alpha'}} \sum_{N_0 L_0} \sum_{i,i'} \delta_{T_{ab}T_{12}} \delta_{T'_{ab}T'_{12}} $$

T coefficients: Jacobi -> (abc)JT state

$$ \times T^{\tilde{a}'\tilde{b}'\tilde{c}'J'_{ab}J}_{N'_{12}N'_3\alpha' N_0 L_0} T^{\tilde{a}\tilde{b}\tilde{c}J_{ab}J}_{N_{12}N_3\alpha N_0 L_0} $$

M coefficients: Antisymmetry

$$ \times M^{i'}_{N'_{12}N'_3\alpha'} M^{i}_{N_{12}N_3\alpha} $$

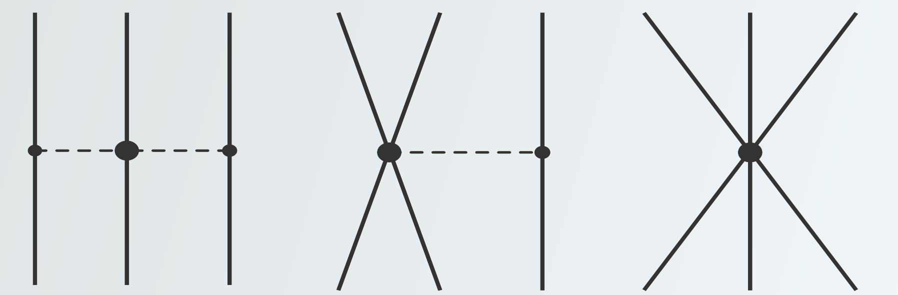
3BME in Jacobi basis

$$ \times \langle N'i'J_{12,3}T, N_0L_0; J | V_{3\mathrm{N}} | NiJ_{12,3}T, N_0L_0; J\rangle $$

$$ V^{2\pi-exch}_{3N} = \frac{g_A^2}{8f_\pi^2} \sum_{i\neq j\neq k} \frac{\sigma_i\cdot\mathbf{Q}_i}{Q_i^2+M_\pi^2} \frac{\sigma_j\cdot\mathbf{Q}'_j}{Q'^2_j+M_\pi^2} \times F^{\alpha\beta}_{ijk}\tau_i^\alpha\tau_j^\beta $$

$$ F^{\alpha\beta}_{ijk}\tau_i^\alpha\tau_j^\beta = \delta_{\alpha\beta}[-4c_1m_\pi^2 + 2c_3\mathbf{q}_i\cdot\mathbf{q}_j] + c_4\epsilon_{\alpha\beta\gamma}\sigma_k\cdot(\mathbf{q}_i\times\mathbf{q}_j) $$

Calculation of TPE is complex



$V^{2\pi-exch}_{3N}$ $V^{1\pi+contact}_{3N}$ $V^{contact}_{3N}$

Computation challenge

$$
\begin{aligned}
&\langle \alpha'; J'_{12} J' T'_{12} T' | W_1^{2\pi\_c_4} | \alpha; J_{12} J T_{12} T \rangle \\
&= c_4 \frac{(72\pi)^2}{(2\pi)^6} \frac{g_A^2}{F_\pi^4} \frac{1}{(\sqrt{3})^3} (-1)^{J_{12}+j'_3+J+T_{12}+\frac{1}{2}+T} (i)^{l'_{12}+l'_3+l_{12}+l_3} \delta_{J'J} \delta_{T'T} \\
&\times \hat{J}'_{12} \hat{J}_{12} \hat{S}'_{12} \hat{S}_{12} \hat{j}'_3 \hat{j}_3 \hat{T}'_{12} \hat{T}_{12} \begin{Bmatrix} T'_{12} & T_{12} & 1 \\ \frac{1}{2} & \frac{1}{2} & T \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ T'_{12} & T_{12} & 1 \end{Bmatrix} \sum_{s_0} \hat{s}_0^2 (-1)^{s_0} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ S'_{12} & S_{12} & s_0 \end{Bmatrix} \\
&\times \sum_{s_1} \hat{s}_1 \langle 1010 | s_1 0 \rangle \begin{Bmatrix} s_0 & s_1 & 1 \\ 1 & 1 & 1 \end{Bmatrix} \sum_{s_2} \hat{s}_2 \langle 1010 | s_2 0 \rangle \\
&\times \sum_{\lambda_1+\lambda_2=s_1} (\frac{1}{\sqrt{2}})^{\lambda_1} (\frac{1}{\sqrt{6}})^{\lambda_2} \hat{\lambda}_1 \sqrt{C_{2s_1+1}^{2\lambda_1+1}} \sum_{\lambda_3+\lambda_4=s_2} (\sqrt{\frac{2}{3}})^{\lambda_3} (-\sqrt{\frac{2}{3}})^{\lambda_4} \hat{\lambda}_3 \sqrt{C_{2s_2+1}^{2\lambda_3+1}} \\
&\times \sum_{\substack{\lambda_i+\lambda_j=\lambda_1 \\ \lambda_k+\lambda_l=\lambda_2}} \hat{\lambda}_k \hat{\lambda}_i \sqrt{C_{2\lambda_1+1}^{2\lambda_i+1} C_{2\lambda_2+1}^{2\lambda_k+1}} (-1)^{\lambda_j+\lambda_l} \\
&\times \sum_{\bar{l},\bar{l}_{12},\bar{l}_3=0}^{\infty} (-1)^{\bar{l}} \sum_{i_1+i_2=\bar{l}} \hat{i}_1 \sqrt{C_{2\bar{l}+1}^{2i_1+1}} \sum_{i_3+i_4=\bar{l}} \hat{i}_3 \sqrt{C_{2\bar{l}+1}^{2i_3+1}} \\
&\times \int dk'_{12} dk_{12} (k'_{12})^{\lambda_i+i_1+1} (k_{12})^{\lambda_j+i_2+1} P_{n'_{12}l'_{12}}(k'_{12}) P_{n_{12}l_{12}}(k_{12}) \\
&\times \int dk'_3 dk_3 (k'_3)^{\lambda_k+\lambda_3+i_3+1} (k_3)^{\lambda_l+\lambda_4+i_4+1} P_{n'_3 l'_3}(k'_3) P_{n_3 l_3}(k_3) F' F g_{\bar{l}\bar{l}_{12}\bar{l}_3}^{s_1 s_2}(k'_{12}, k_{12}, k'_3, k_3) \\
&\times \sum_{s_3} \hat{s}_3^2 \begin{Bmatrix} s_0 & \lambda_1 & s_3 \\ \lambda_2 & 1 & s_1 \end{Bmatrix} \sum_{s_4} \hat{s}_4^2 (-1)^{s_4} \begin{Bmatrix} 1 & s_2 & 1 \\ \lambda_2 & s_3 & s_4 \end{Bmatrix} \\
&\times \sum_{\lambda_{3k}\lambda_{4l}} \hat{\lambda}_{3k} \hat{\lambda}_{4l} \begin{Bmatrix} \lambda_3 & \lambda_4 & s_2 \\ \lambda_k & \lambda_l & \lambda_2 \\ \lambda_{3k} & \lambda_{4l} & s_4 \end{Bmatrix} \langle \lambda_3 0 \lambda_k 0 | \lambda_{3k} 0 \rangle \langle \lambda_4 0 \lambda_l 0 | \lambda_{4l} 0 \rangle \\
&\times \sum_{q_1 q_2} \hat{q}_1 \hat{q}_2 (-1)^{q_2} \begin{Bmatrix} q_1 & q_2 & \bar{l} \\ i_2 & i_1 & \bar{l}_{12} \end{Bmatrix} \langle i_1 0 \bar{l}_{12} 0 | q_1 0 \rangle \langle i_2 0 \bar{l}_{12} 0 | q_2 0 \rangle \langle \lambda_i 0 q_1 0 | l'_{12} 0 \rangle \langle \lambda_j 0 q_2 0 | l_{12} 0 \rangle \\
&\times \sum_{q_3 q_4} \hat{q}_3 \hat{q}_4 (-1)^{q_4} \begin{Bmatrix} q_3 & q_4 & \bar{l} \\ i_4 & i_3 & \bar{l}_3 \end{Bmatrix} \langle i_3 0 \bar{l}_3 0 | q_3 0 \rangle \langle i_4 0 \bar{l}_3 0 | q_4 0 \rangle \langle \lambda_{3k} 0 q_3 0 | l'_3 0 \rangle \langle \lambda_{4l} 0 q_4 0 | l_3 0 \rangle \\
&\times \sum_{t_0} \hat{t}_0^2 (-1)^{t_0} \begin{Bmatrix} J'_{12} & j'_3 & J \\ j_3 & J_{12} & t_0 \end{Bmatrix} \sum_{t_1} \hat{t}_1^2 (-1)^{t_1} \begin{Bmatrix} \lambda_1 & s_0 & s_3 \\ t_0 & \bar{l} & t_1 \end{Bmatrix} \begin{Bmatrix} \lambda_i & \lambda_j & \lambda_1 \\ q_1 & q_2 & \bar{l} \\ l'_{12} & l_{12} & t_1 \end{Bmatrix} \begin{Bmatrix} l'_{12} & l_{12} & t_1 \\ S'_{12} & S_{12} & s_0 \\ J'_{12} & J_{12} & t_0 \end{Bmatrix} \\
&\times \sum_{t_4} \hat{t}_4^2 (-1)^{t_4} \begin{Bmatrix} s_4 & 1 & s_3 \\ t_0 & \bar{l} & t_4 \end{Bmatrix} \begin{Bmatrix} \lambda_{3k} & \lambda_{4l} & s_4 \\ q_3 & q_4 & \bar{l} \\ l'_3 & l'_3 & t_4 \end{Bmatrix} \begin{Bmatrix} l'_3 & l_3 & t_4 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j'_3 & j_3 & t_0 \end{Bmatrix} \delta_{l'_{12}t_2} \delta_{l_{12}t_3} \delta_{l'_3 t_5} \delta_{l_3 t_6}.
\end{aligned}
\tag{190}
$$

$$ {}_A\langle [[\bullet\bullet]\bullet]_{JT} | V_{3N} | [[\bullet\bullet]\bullet]_{JT} \rangle_A $$

1. Complexity

Computation challenge

$$
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&\langle \alpha'; J'_{12}J'T'_{12}T' | W_1^{2\pi\_c_4} | \alpha; J_{12}JT_{12}T \rangle \\
&= c_4 \frac{(72\pi)^2}{(2\pi)^6} \frac{g_A^2}{F_\pi^4} \frac{1}{(\sqrt{3})^3} (-1)^{J_{12}+j'_3+J+T_{12}+\frac{1}{2}+T} (i)^{l'_{12}+l'_3+l_{12}+l_3} \delta_{J'J}\delta_{T'T} \\
&\times \hat{J}'_{12}\hat{J}_{12}\hat{S}'_{12}\hat{S}_{12}\hat{j}'_3\hat{j}_3\hat{T}'_{12}\hat{T}_{12} \begin{Bmatrix} T'_{12} & T_{12} & 1 \\ \frac{1}{2} & \frac{1}{2} & T \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ T'_{12} & T_{12} & 1 \end{Bmatrix} \sum_{s_0} \hat{s}_0^2 (-1)^{s_0} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ S'_{12} & S_{12} & s_0 \end{Bmatrix} \\
&\times \sum_{s_1} \hat{s}_1 \langle 1010|s_1 0\rangle \begin{Bmatrix} s_0 & s_1 & 1 \\ 1 & 1 & 1 \end{Bmatrix} \sum_{s_2} \hat{s}_2 \langle 1010|s_2 0\rangle \\
&\times \sum_{\lambda_1+\lambda_2=s_1} (\frac{1}{\sqrt{2}})^{\lambda_1} (\frac{1}{\sqrt{6}})^{\lambda_2} \hat{\lambda}_1 \sqrt{C_{2s_1+1}^{2\lambda_1+1}} \sum_{\lambda_3+\lambda_4=s_2} (\sqrt{\frac{2}{3}})^{\lambda_3} (-\sqrt{\frac{2}{3}})^{\lambda_4} \hat{\lambda}_3 \sqrt{C_{2s_2+1}^{2\lambda_3+1}} \\
&\times \sum_{\substack{\lambda_i+\lambda_j=\lambda_1 \\ \lambda_k+\lambda_l=\lambda_2}} \hat{\lambda}_k \hat{\lambda}_i \sqrt{C_{2\lambda_1+1}^{2\lambda_i+1} C_{2\lambda_2+1}^{2\lambda_k+1}} (-1)^{\lambda_j+\lambda_l} \\
&\times \sum_{\bar{l},\bar{l}_{12},\bar{l}_3=0}^{\infty} (-1)^{\bar{l}} \sum_{i_1+i_2=\bar{l}} \hat{i}_1 \sqrt{C_{2\bar{l}+1}^{2i_1+1}} \sum_{i_3+i_4=\bar{l}} \hat{i}_3 \sqrt{C_{2\bar{l}+1}^{2i_3+1}} \\
&\times \int dk'_{12} dk_{12} (k'_{12})^{\lambda_i+i_1+1} (k_{12})^{\lambda_j+i_2+1} P_{n'_{12}l'_{12}}(k'_{12}) P_{n_{12}l_{12}}(k_{12}) \\
&\times \int dk'_3 dk_3 (k'_3)^{\lambda_k+\lambda_3+i_3+1} (k_3)^{\lambda_l+\lambda_4+i_4+1} P_{n'_3l'_3}(k'_3) P_{n_3l_3}(k_3) F' F g^{s_1s_2}_{\bar{l}\bar{l}_{12}\bar{l}_3}(k'_{12},k_{12},k'_3,k_3) \\
&\times \sum_{s_3} \hat{s}_3^2 \begin{Bmatrix} s_0 & \lambda_1 & s_3 \\ \lambda_2 & 1 & s_1 \end{Bmatrix} \sum_{s_4} \hat{s}_4^2 (-1)^{s_4} \begin{Bmatrix} 1 & s_2 & 1 \\ \lambda_2 & s_3 & s_4 \end{Bmatrix} \\
&\times \sum_{\lambda_{3k}\lambda_{4l}} \hat{\lambda}_{3k} \hat{\lambda}_{4l} \begin{Bmatrix} \lambda_3 & \lambda_4 & s_2 \\ \lambda_k & \lambda_l & \lambda_2 \\ \lambda_{3k} & \lambda_{4l} & s_4 \end{Bmatrix} \langle \lambda_3 0 \lambda_k 0 | \lambda_{3k} 0\rangle \langle \lambda_4 0 \lambda_l 0 | \lambda_{4l} 0 \rangle \\
&\times \sum_{q_1q_2} \hat{q}_1 \hat{q}_2 (-1)^{q_2} \begin{Bmatrix} q_1 & q_2 & \bar{l} \\ i_2 & i_1 & \bar{l}_{12} \end{Bmatrix} \langle i_1 0 \bar{l}_{12} 0 | q_1 0\rangle \langle i_2 0 \bar{l}_{12} 0 | q_2 0 \rangle \langle \lambda_i 0 q_1 0 | l'_{12} 0\rangle \langle \lambda_j 0 q_2 0 | l_{12} 0 \rangle \\
&\times \sum_{q_3q_4} \hat{q}_3 \hat{q}_4 (-1)^{q_4} \begin{Bmatrix} q_3 & q_4 & \bar{l} \\ i_4 & i_3 & \bar{l}_3 \end{Bmatrix} \langle i_3 0 \bar{l}_3 0 | q_3 0\rangle \langle i_4 0 \bar{l}_3 0 | q_4 0 \rangle \langle \lambda_{3k} 0 q_3 0 | l'_3 0\rangle \langle \lambda_{4l} 0 q_4 0 | l_3 0 \rangle \\
&\times \sum_{t_0} \hat{t}_0^2 (-1)^{t_0} \begin{Bmatrix} J'_{12} & j'_3 & J \\ j_3 & J_{12} & t_0 \end{Bmatrix} \sum_{t_1} \hat{t}_1^2 (-1)^{t_1} \begin{Bmatrix} \lambda_1 & s_0 & s_3 \\ t_0 & \bar{l} & t_1 \end{Bmatrix} \begin{Bmatrix} \lambda_i & \lambda_j & \lambda_1 \\ q_1 & q_2 & \bar{l} \\ l'_{12} & l_{12} & t_1 \end{Bmatrix} \begin{Bmatrix} l'_{12} & l_{12} & t_1 \\ S'_{12} & S_{12} & s_0 \\ J'_{12} & J_{12} & t_0 \end{Bmatrix} \\
&\times \sum_{t_4} \hat{t}_4^2 (-1)^{t_4} \begin{Bmatrix} s_4 & 1 & s_3 \\ t_0 & \bar{l} & t_4 \end{Bmatrix} \begin{Bmatrix} \lambda_{3k} & \lambda_{4l} & s_4 \\ q_3 & q_4 & \bar{l} \\ l'_3 & l'_3 & t_4 \end{Bmatrix} \begin{Bmatrix} l'_3 & l_3 & t_4 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j'_3 & j_3 & t_0 \end{Bmatrix} \delta_{l'_{12}t_2} \delta_{l_{12}t_3} \delta_{l'_3t_5} \delta_{l_3t_6}.
\end{aligned}
\tag{190}
$$

$${}_A\langle [[\bullet\bullet]\bullet]_{JT} | V_{3N} | [[\bullet\bullet]\bullet]_{JT} \rangle_A$$

1. Complexity

for each **configuration**:

Computation challenge

$$
\begin{aligned}
&\langle \alpha'; J'_{12}J'T'_{12}T' | W_1^{2\pi - c_4} | \alpha; J_{12}JT_{12}T\rangle \\
&= c_4 \frac{(72\pi)^2}{(2\pi)^6}\frac{g_A^2}{F_\pi^4}\frac{1}{(\sqrt{3})^3}(-1)^{J_{12}+j'_3+J+T_{12}+\frac{1}{2}+T}(i)^{l'_{12}+l'_3+l_{12}+l_3}\delta_{J'J}\delta_{T'T} \\
&\times \hat{J}'_{12}\hat{J}_{12}\hat{S}'_{12}\hat{S}_{12}\hat{j}'_3\hat{j}_3\hat{T}'_{12}\hat{T}_{12}\begin{Bmatrix} T'_{12} & T_{12} & 1 \\ \frac{1}{2} & \frac{1}{2} & T \end{Bmatrix}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ T'_{12} & T_{12} & 1 \end{Bmatrix}\sum_{s_0}\hat{s}_0^2(-1)^{s_0}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ S'_{12} & S_{12} & s_0 \end{Bmatrix} \\
&\times \sum_{s_1}\hat{s}_1\langle 1010|s_10\rangle\begin{Bmatrix} s_0 & s_1 & 1 \\ 1 & 1 & 1 \end{Bmatrix}\sum_{s_2}\hat{s}_2\langle 1010|s_20\rangle \\
&\times \sum_{\lambda_1+\lambda_2=s_1}\left(\frac{1}{\sqrt{2}}\right)^{\lambda_1}\left(\frac{1}{\sqrt{6}}\right)^{\lambda_2}\hat{\lambda}_1\sqrt{C_{2s_1+1}^{2\lambda_1+1}}\sum_{\lambda_3+\lambda_4=s_2}\left(\sqrt{\frac{2}{3}}\right)^{\lambda_3}\left(-\sqrt{\frac{2}{3}}\right)^{\lambda_4}\hat{\lambda}_3\sqrt{C_{2s_2+1}^{2\lambda_3+1}} \\
&\times \sum_{\substack{\lambda_i+\lambda_j=\lambda_1 \\ \lambda_k+\lambda_l=\lambda_2}}\hat{\lambda}_k\hat{\lambda}_i\sqrt{C_{2\lambda_1+1}^{2\lambda_i+1}C_{2\lambda_2+1}^{2\lambda_k+1}}(-1)^{\lambda_j+\lambda_l} \\
&\times \sum_{\bar{l}_{12},\bar{l}_3=0}^{\infty}(-1)^{\bar{l}}\sum_{i_1+i_2=\bar{l}}\hat{i}_1\sqrt{C_{2\bar{l}+1}^{2i_1+1}}\sum_{i_3+i_4=\bar{l}}\hat{i}_3\sqrt{C_{2\bar{l}+1}^{2i_3+1}} \\
&\times \int dk'_{12}dk_{12}(k'_{12})^{\lambda_i+i_1+1}(k_{12})^{\lambda_j+i_2+1}P_{n'_{12}l'_{12}}(k'_{12})P_{n_{12}l_{12}}(k_{12}) \\
&\times \int dk'_3dk_3(k'_3)^{\lambda_k+\lambda_3+i_3+1}(k_3)^{\lambda_l+\lambda_4+i_4+1}P_{n'_3l'_3}(k'_3)P_{n_3l_3}(k_3)F'Fg_{\bar{l}\bar{l}_{12}\bar{l}_3}^{s_1s_2}(k'_{12},k_{12},k'_3,k_3) \\
&\times \sum_{s_3}\hat{s}_3^2\begin{Bmatrix} s_0 & \lambda_1 & s_3 \\ \lambda_2 & 1 & s_1 \end{Bmatrix}\sum_{s_4}\hat{s}_4^2(-1)^{s_4}\begin{Bmatrix} 1 & s_2 & 1 \\ \lambda_2 & s_3 & s_4 \end{Bmatrix} \\
&\times \sum_{\lambda_{3k}\lambda_{4l}}\hat{\lambda}_{3k}\hat{\lambda}_{4l}\begin{Bmatrix} \lambda_3 & \lambda_4 & s_2 \\ \lambda_k & \lambda_l & \lambda_2 \\ \lambda_{3k} & \lambda_{4l} & s_4 \end{Bmatrix}\langle\lambda_30\lambda_k0|\lambda_{3k}0\rangle\langle\lambda_40\lambda_l0|\lambda_{4l}0\rangle \\
&\times \sum_{q_1q_2}\hat{q}_1\hat{q}_2(-1)^{q_2}\begin{Bmatrix} q_1 & q_2 & \bar{l} \\ i_2 & i_1 & \bar{l}_{12} \end{Bmatrix}\langle i_10\bar{l}_{12}0|q_10\rangle\langle i_20\bar{l}_{12}0|q_20\rangle\langle\lambda_i0q_10|l'_{12}0\rangle\langle\lambda_j0q_20|l_{12}0\rangle \\
&\times \sum_{q_3q_4}\hat{q}_3\hat{q}_4(-1)^{q_4}\begin{Bmatrix} q_3 & q_4 & \bar{l} \\ i_4 & i_3 & \bar{l}_3 \end{Bmatrix}\langle i_30\bar{l}_30|q_30\rangle\langle i_40\bar{l}_30|q_40\rangle\langle\lambda_{3k}0q_30|l'_30\rangle\langle\lambda_{4l}0q_40|l_30\rangle \\
&\times \sum_{t_0}\hat{t}_0^2(-1)^{t_0}\begin{Bmatrix} J'_{12} & j'_3 & J \\ j_3 & J_{12} & t_0 \end{Bmatrix}\sum_{t_1}\hat{t}_1^2(-1)^{t_1}\begin{Bmatrix} \lambda_1 & s_0 & s_3 \\ t_0 & \bar{l} & t_1 \end{Bmatrix}\begin{Bmatrix} \lambda_i & \lambda_j & \lambda_1 \\ q_1 & q_2 & \bar{l} \\ l'_{12} & l_{12} & t_1 \end{Bmatrix}\begin{Bmatrix} l'_{12} & l_{12} & t_1 \\ S'_{12} & S_{12} & s_0 \\ J'_{12} & J_{12} & t_0 \end{Bmatrix} \\
&\times \sum_{t_4}\hat{t}_4^2(-1)^{t_4}\begin{Bmatrix} s_4 & 1 & s_3 \\ t_0 & \bar{l} & t_4 \end{Bmatrix}\begin{Bmatrix} \lambda_{3k} & \lambda_{4l} & s_4 \\ q_3 & q_4 & \bar{l} \\ l'_3 & l'_3 & t_4 \end{Bmatrix}\begin{Bmatrix} l'_3 & l_3 & t_4 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j'_3 & j_3 & t_0 \end{Bmatrix}\delta_{l'_{12}t_2}\delta_{l_{12}t_3}\delta_{l'_3t_5}\delta_{l_3t_6}.
\end{aligned}
\tag{190}
$$

$$
{}_A\langle [[\bullet\bullet]\bullet]_{JT} | V_{3N} | [[\bullet\bullet]\bullet]_{JT}\rangle_A
$$

1. Complexity

for each **configuration**:

- **Summation**

Computation challenge

$$
\begin{aligned}
&\langle \alpha'; J'_{12}J'T'_{12}T' | W_1^{2\pi - c_4} | \alpha; J_{12}JT_{12}T \rangle \\
&= c_4 \frac{(72\pi)^2}{(2\pi)^6} \frac{g_A^2}{F_\pi^4} \frac{1}{(\sqrt{3})^3} (-1)^{J_{12}+j'_3+J+T_{12}+\frac{1}{2}+T} (i)^{l'_{12}+l'_3+l_{12}+l_3} \delta_{J'J}\delta_{T'T} \\
&\times \hat{J}'_{12}\hat{J}_{12}\hat{S}'_{12}\hat{S}_{12}\hat{j}'_3\hat{j}_3\hat{T}'_{12}\hat{T}_{12} \begin{Bmatrix} T'_{12} & T_{12} & 1 \\ \frac{1}{2} & \frac{1}{2} & T \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ T'_{12} & T_{12} & 1 \end{Bmatrix} \sum_{s_0} \hat{s}_0^2 (-1)^{s_0} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ S'_{12} & S_{12} & s_0 \end{Bmatrix} \\
&\times \sum_{s_1} \hat{s}_1 \langle 1010|s_1 0\rangle \begin{Bmatrix} s_0 & s_1 & 1 \\ 1 & 1 & 1 \end{Bmatrix} \sum_{s_2} \hat{s}_2 \langle 1010|s_2 0\rangle \\
&\times \sum_{\lambda_1+\lambda_2=s_1} \left(\frac{1}{\sqrt{2}}\right)^{\lambda_1} \left(\frac{1}{\sqrt{6}}\right)^{\lambda_2} \hat{\lambda}_1 \sqrt{C_{2s_1+1}^{2\lambda_1+1}} \sum_{\lambda_3+\lambda_4=s_2} \left(\sqrt{\frac{2}{3}}\right)^{\lambda_3} \left(-\sqrt{\frac{2}{3}}\right)^{\lambda_4} \hat{\lambda}_3 \sqrt{C_{2s_2+1}^{2\lambda_3+1}} \\
&\times \sum_{\substack{\lambda_i+\lambda_j=\lambda_1 \\ \lambda_k+\lambda_l=\lambda_2}} \hat{\lambda}_k \hat{\lambda}_i \sqrt{C_{2\lambda_1+1}^{2\lambda_i+1} C_{2\lambda_2+1}^{2\lambda_k+1}} (-1)^{\lambda_j+\lambda_l} \\
&\times \sum_{\bar{l}_{12},\bar{l}_3=0}^{\infty} (-1)^{\bar{l}} \sum_{i_1+i_2=\bar{l}} \hat{i}_1 \sqrt{C_{2\bar{l}+1}^{2i_1+1}} \sum_{i_3+i_4=\bar{l}} \hat{i}_3 \sqrt{C_{2\bar{l}+1}^{2i_3+1}} \\
&\times \int dk'_{12} dk_{12} (k'_{12})^{\lambda_i+i_1+1} (k_{12})^{\lambda_j+i_2+1} P_{n'_{12}l'_{12}}(k'_{12}) P_{n_{12}l_{12}}(k_{12}) \\
&\times \int dk'_3 dk_3 (k'_3)^{\lambda_k+\lambda_3+i_3+1} (k_3)^{\lambda_l+\lambda_4+i_4+1} P_{n'_3l'_3}(k'_3) P_{n_3l_3}(k_3) F' F g^{s_1 s_2}_{\bar{l}\bar{l}_{12}\bar{l}_3}(k'_{12},k_{12},k'_3,k_3) \\
&\times \sum_{s_3} \hat{s}_3^2 \begin{Bmatrix} s_0 & \lambda_1 & s_3 \\ \lambda_2 & 1 & s_1 \end{Bmatrix} \sum_{s_4} \hat{s}_4^2 (-1)^{s_4} \begin{Bmatrix} 1 & s_2 & 1 \\ \lambda_2 & s_3 & s_4 \end{Bmatrix} \\
&\times \sum_{\lambda_{3k}\lambda_{4l}} \hat{\lambda}_{3k} \hat{\lambda}_{4l} \begin{Bmatrix} \lambda_3 & \lambda_4 & s_2 \\ \lambda_k & \lambda_l & \lambda_2 \\ \lambda_{3k} & \lambda_{4l} & s_4 \end{Bmatrix} \langle \lambda_3 0 \lambda_k 0 | \lambda_{3k} 0\rangle \langle \lambda_4 0 \lambda_l 0 | \lambda_{4l} 0\rangle \\
&\times \sum_{q_1 q_2} \hat{q}_1 \hat{q}_2 (-1)^{q_2} \begin{Bmatrix} q_1 & q_2 & \bar{l} \\ i_2 & i_1 & \bar{l}_{12} \end{Bmatrix} \langle i_1 0 \bar{l}_{12} 0 | q_1 0\rangle \langle i_2 0 \bar{l}_{12} 0 | q_2 0\rangle \langle \lambda_i 0 q_1 0 | l'_{12} 0\rangle \langle \lambda_j 0 q_2 0 | l_{12} 0\rangle \\
&\times \sum_{q_3 q_4} \hat{q}_3 \hat{q}_4 (-1)^{q_4} \begin{Bmatrix} q_3 & q_4 & \bar{l} \\ i_4 & i_3 & \bar{l}_3 \end{Bmatrix} \langle i_3 0 \bar{l}_3 0 | q_3 0\rangle \langle i_4 0 \bar{l}_3 0 | q_4 0\rangle \langle \lambda_{3k} 0 q_3 0 | l'_3 0\rangle \langle \lambda_{4l} 0 q_4 0 | l_3 0\rangle \\
&\times \sum_{t_0} \hat{t}_0^2 (-1)^{t_0} \begin{Bmatrix} J'_{12} & j'_3 & J \\ j_3 & J_{12} & t_0 \end{Bmatrix} \sum_{t_1} \hat{t}_1^2 (-1)^{t_1} \begin{Bmatrix} \lambda_1 & s_0 & s_3 \\ t_0 & \bar{l} & t_1 \end{Bmatrix} \begin{Bmatrix} \lambda_i & \lambda_j & \lambda_1 \\ q_1 & q_2 & \bar{l} \\ l'_{12} & l_{12} & t_1 \end{Bmatrix} \begin{Bmatrix} l'_{12} & l_{12} & t_1 \\ S'_{12} & S_{12} & s_0 \\ J'_{12} & J_{12} & t_0 \end{Bmatrix} \\
&\times \sum_{t_4} \hat{t}_4^2 (-1)^{t_4} \begin{Bmatrix} s_4 & 1 & s_3 \\ t_0 & \bar{l} & t_4 \end{Bmatrix} \begin{Bmatrix} \lambda_{3k} & \lambda_{4l} & s_4 \\ q_3 & q_4 & \bar{l} \\ l'_3 & l'_3 & t_4 \end{Bmatrix} \begin{Bmatrix} l'_3 & l_3 & t_4 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j'_3 & j_3 & t_0 \end{Bmatrix} \delta_{l'_{12}t_2} \delta_{l_{12}t_3} \delta_{l'_3 t_5} \delta_{l_3 t_6}. \qquad (190)
\end{aligned}
$$

$$ {}_A\langle [[\bullet\bullet]\bullet]_{JT} | V_{3N} | [[\bullet\bullet]\bullet]_{JT} \rangle_A $$

1. Complexity

for each **configuration**:

- **Summation**

  different range by configuration

Computation challenge

$$
\begin{aligned}
&\langle \alpha'; J'_{12}J'T'_{12}T'|W_1^{2\pi-c_4}|\alpha; J_{12}JT_{12}T\rangle \\
&= c_4\frac{(72\pi)^2}{(2\pi)^6}\frac{g_A^2}{F_\pi^4}\frac{1}{(\sqrt{3})^3}(-1)^{J_{12}+j_3'+J+T_{12}+\frac{1}{2}+T}(i)^{l'_{12}+l'_3+l_{12}+l_3}\delta_{J'J}\delta_{T'T} \\
&\times \hat{J}'_{12}\hat{J}_{12}\hat{S}'_{12}\hat{S}_{12}\hat{j}'_3\hat{j}_3\hat{T}'_{12}\hat{T}_{12}\begin{Bmatrix} T'_{12} & T_{12} & 1 \\ \frac{1}{2} & \frac{1}{2} & T \end{Bmatrix}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ T'_{12} & T_{12} & 1 \end{Bmatrix}\sum_{s_0}\hat{s}_0^2(-1)^{s_0}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ S'_{12} & S_{12} & s_0 \end{Bmatrix} \\
&\times \sum_{s_1}\hat{s}_1\langle 1010|s_10\rangle\begin{Bmatrix} s_0 & s_1 & 1 \\ 1 & 1 & 1 \end{Bmatrix}\sum_{s_2}\hat{s}_2\langle 1010|s_20\rangle \\
&\times \sum_{\lambda_1+\lambda_2=s_1}\left(\frac{1}{\sqrt{2}}\right)^{\lambda_1}\left(\frac{1}{\sqrt{6}}\right)^{\lambda_2}\hat{\lambda}_1\sqrt{C_{2s_1+1}^{2\lambda_1+1}}\sum_{\lambda_3+\lambda_4=s_2}\left(\sqrt{\frac{2}{3}}\right)^{\lambda_3}\left(-\sqrt{\frac{2}{3}}\right)^{\lambda_4}\hat{\lambda}_3\sqrt{C_{2s_2+1}^{2\lambda_3+1}} \\
&\times \sum_{\substack{\lambda_i+\lambda_j=\lambda_1 \\ \lambda_k+\lambda_l=\lambda_2}}\hat{\lambda}_k\hat{\lambda}_i\sqrt{C_{2\lambda_1+1}^{2\lambda_i+1}C_{2\lambda_2+1}^{2\lambda_k+1}}(-1)^{\lambda_j+\lambda_l} \\
&\times \sum_{\bar{l}_{12},\bar{l}_3=0}^{\infty}(-1)^{\bar{l}}\sum_{i_1+i_2=\bar{l}}\hat{i}_1\sqrt{C_{2\bar{l}+1}^{2i_1+1}}\sum_{i_3+i_4=\bar{l}}\hat{i}_3\sqrt{C_{2\bar{l}+1}^{2i_3+1}} \\
&\times \int dk'_{12}dk_{12}(k'_{12})^{\lambda_i+i_1+1}(k_{12})^{\lambda_j+i_2+1}P_{n'_{12}l'_{12}}(k'_{12})P_{n_{12}l_{12}}(k_{12}) \\
&\times \int dk'_3dk_3(k'_3)^{\lambda_k+\lambda_3+i_3+1}(k_3)^{\lambda_l+\lambda_4+i_4+1}P_{n'_3l'_3}(k'_3)P_{n_3l_3}(k_3)F'Fg^{s_1s_2}_{\bar{l}\bar{l}_{12}\bar{l}_3}(k'_{12},k_{12},k'_3,k_3) \\
&\times \sum_{s_3}\hat{s}_3^2\begin{Bmatrix} s_0 & \lambda_1 & s_3 \\ \lambda_2 & 1 & s_1 \end{Bmatrix}\sum_{s_4}\hat{s}_4^2(-1)^{s_4}\begin{Bmatrix} 1 & s_2 & 1 \\ \lambda_2 & s_3 & s_4 \end{Bmatrix} \\
&\times \sum_{\lambda_{3k}\lambda_{4l}}\hat{\lambda}_{3k}\hat{\lambda}_{4l}\begin{Bmatrix} \lambda_3 & \lambda_4 & s_2 \\ \lambda_k & \lambda_l & \lambda_2 \\ \lambda_{3k} & \lambda_{4l} & s_4 \end{Bmatrix}\langle\lambda_30\lambda_k0|\lambda_{3k}0\rangle\langle\lambda_40\lambda_l0|\lambda_{4l}0\rangle \\
&\times \sum_{q_1q_2}\hat{q}_1\hat{q}_2(-1)^{q_2}\begin{Bmatrix} q_1 & q_2 & \bar{l} \\ i_2 & i_1 & \bar{l}_{12} \end{Bmatrix}\langle i_10\bar{l}_{12}0|q_10\rangle\langle i_20\bar{l}_{12}0|q_20\rangle\langle\lambda_i0q_10|l'_{12}0\rangle\langle\lambda_j0q_20|l_{12}0\rangle \\
&\times \sum_{q_3q_4}\hat{q}_3\hat{q}_4(-1)^{q_4}\begin{Bmatrix} q_3 & q_4 & \bar{l} \\ i_4 & i_3 & \bar{l}_3 \end{Bmatrix}\langle i_30\bar{l}_30|q_30\rangle\langle i_40\bar{l}_30|q_40\rangle\langle\lambda_{3k}0q_30|l'_30\rangle\langle\lambda_{4l}0q_40|l_30\rangle \\
&\times \sum_{t_0}\hat{t}_0^2(-1)^{t_0}\begin{Bmatrix} J'_{12} & j'_3 & J \\ j_3 & J_{12} & t_0 \end{Bmatrix}\sum_{t_1}\hat{t}_1^2(-1)^{t_1}\begin{Bmatrix} \lambda_1 & s_0 & s_3 \\ t_0 & \bar{l} & t_1 \end{Bmatrix}\begin{Bmatrix} \lambda_i & \lambda_j & \lambda_1 \\ q_1 & q_2 & \bar{l} \\ l'_{12} & l_{12} & t_1 \end{Bmatrix}\begin{Bmatrix} l'_{12} & l_{12} & t_1 \\ S'_{12} & S_{12} & s_0 \\ J'_{12} & J_{12} & t_0 \end{Bmatrix} \\
&\times \sum_{t_4}\hat{t}_4^2(-1)^{t_4}\begin{Bmatrix} s_4 & 1 & s_3 \\ t_0 & \bar{l} & t_4 \end{Bmatrix}\begin{Bmatrix} \lambda_{3k} & \lambda_{4l} & s_4 \\ q_3 & q_4 & \bar{l} \\ l'_3 & l'_3 & t_4 \end{Bmatrix}\begin{Bmatrix} l'_3 & l_3 & t_4 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j'_3 & j_3 & t_0 \end{Bmatrix}\delta_{l'_{12}t_2}\delta_{l_{12}t_3}\delta_{l'_3t_5}\delta_{l_3t_6}.
\end{aligned}
\tag{190}
$$

$$ {}_A\langle[[\bullet\bullet]\bullet]_{JT}|V_{3N}|[[\bullet\bullet]\bullet]_{JT}\rangle_A $$

1. Complexity

for each **configuration**:

- **Summation**

  different range by configuration

- **Integration**

Computation challenge

$$
\begin{aligned}
&\langle \alpha'; J'_{12} J' T'_{12} T' | W_1^{2\pi\_c_4} | \alpha; J_{12} J T_{12} T \rangle \\
&= c_4 \frac{(72\pi)^2}{(2\pi)^6} \frac{g_A^2}{F_\pi^4} \frac{1}{(\sqrt{3})^3} (-1)^{J_{12}+j'_3+J+T_{12}+\frac{1}{2}+T} (i)^{l'_{12}+l'_3+l_{12}+l_3} \delta_{J'J}\delta_{T'T} \\
&\times \hat{J}'_{12}\hat{J}_{12}\hat{S}'_{12}\hat{S}_{12}\hat{j}'_3\hat{j}_3\hat{T}'_{12}\hat{T}_{12} \begin{Bmatrix} T'_{12} & T_{12} & 1 \\ \frac{1}{2} & \frac{1}{2} & T \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ T'_{12} & T_{12} & 1 \end{Bmatrix} \sum_{s_0} \hat{s}_0^2 (-1)^{s_0} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ S'_{12} & S_{12} & s_0 \end{Bmatrix} \\
&\times \sum_{s_1} \hat{s}_1 \langle 1010 | s_1 0 \rangle \begin{Bmatrix} s_0 & s_1 & 1 \\ 1 & 1 & 1 \end{Bmatrix} \sum_{s_2} \hat{s}_2 \langle 1010 | s_2 0 \rangle \\
&\times \sum_{\lambda_1+\lambda_2=s_1} \left(\frac{1}{\sqrt{2}}\right)^{\lambda_1} \left(\frac{1}{\sqrt{6}}\right)^{\lambda_2} \hat{\lambda}_1 \sqrt{C_{2s_1+1}^{2\lambda_1+1}} \sum_{\lambda_3+\lambda_4=s_2} \left(\sqrt{\frac{2}{3}}\right)^{\lambda_3} \left(-\sqrt{\frac{2}{3}}\right)^{\lambda_4} \hat{\lambda}_3 \sqrt{C_{2s_2+1}^{2\lambda_3+1}} \\
&\times \sum_{\substack{\lambda_i+\lambda_j=\lambda_1 \\ \lambda_k+\lambda_l=\lambda_2}} \hat{\lambda}_k \hat{\lambda}_i \sqrt{C_{2\lambda_1+1}^{2\lambda_i+1} C_{2\lambda_2+1}^{2\lambda_k+1}} (-1)^{\lambda_j+\lambda_l} \\
&\times \sum_{\bar{l}_{12},\bar{l}_3=0}^{\infty} (-1)^{\bar{l}} \sum_{i_1+i_2=\bar{l}} \hat{i}_1 \sqrt{C_{2\bar{l}+1}^{2i_1+1}} \sum_{i_3+i_4=\bar{l}} \hat{i}_3 \sqrt{C_{2\bar{l}+1}^{2i_3+1}} \\
&\times \int dk'_{12} dk_{12} (k'_{12})^{\lambda_i+i_1+1} (k_{12})^{\lambda_j+i_2+1} P_{n'_{12}l'_{12}}(k'_{12}) P_{n_{12}l_{12}}(k_{12}) \\
&\times \int dk'_3 dk_3 (k'_3)^{\lambda_k+\lambda_3+i_3+1} (k_3)^{\lambda_l+\lambda_4+i_4+1} P_{n'_3l'_3}(k'_3) P_{n_3l_3}(k_3) F' F g^{s_1s_2}_{\bar{l}\bar{l}_{12}\bar{l}_3}(k'_{12}, k_{12}, k'_3, k_3) \\
&\times \sum_{s_3} \hat{s}_3^2 \begin{Bmatrix} s_0 & \lambda_1 & s_3 \\ \lambda_2 & 1 & s_1 \end{Bmatrix} \sum_{s_4} \hat{s}_4^2 (-1)^{s_4} \begin{Bmatrix} 1 & s_2 & 1 \\ \lambda_2 & s_3 & s_4 \end{Bmatrix} \\
&\times \sum_{\lambda_{3k}\lambda_{4l}} \hat{\lambda}_{3k} \hat{\lambda}_{4l} \begin{Bmatrix} \lambda_3 & \lambda_4 & s_2 \\ \lambda_k & \lambda_l & \lambda_2 \\ \lambda_{3k} & \lambda_{4l} & s_4 \end{Bmatrix} \langle \lambda_3 0 \lambda_k 0 | \lambda_{3k} 0 \rangle \langle \lambda_4 0 \lambda_l 0 | \lambda_{4l} 0 \rangle \\
&\times \sum_{q_1q_2} \hat{q}_1 \hat{q}_2 (-1)^{q_2} \begin{Bmatrix} q_1 & q_2 & \bar{l} \\ i_2 & i_1 & \bar{l}_{12} \end{Bmatrix} \langle i_1 0 \bar{l}_{12} 0 | q_1 0 \rangle \langle i_2 0 \bar{l}_{12} 0 | q_2 0 \rangle \langle \lambda_i 0 q_1 0 | l'_{12} 0 \rangle \langle \lambda_j 0 q_2 0 | l_{12} 0 \rangle \\
&\times \sum_{q_3q_4} \hat{q}_3 \hat{q}_4 (-1)^{q_4} \begin{Bmatrix} q_3 & q_4 & \bar{l} \\ i_4 & i_3 & \bar{l}_3 \end{Bmatrix} \langle i_3 0 \bar{l}_3 0 | q_3 0 \rangle \langle i_4 0 \bar{l}_3 0 | q_4 0 \rangle \langle \lambda_{3k} 0 q_3 0 | l'_3 0 \rangle \langle \lambda_{4l} 0 q_4 0 | l_3 0 \rangle \\
&\times \sum_{t_0} \hat{t}_0^2 (-1)^{t_0} \begin{Bmatrix} J'_{12} & j'_3 & J \\ j_3 & J_{12} & t_0 \end{Bmatrix} \sum_{t_1} \hat{t}_1^2 (-1)^{t_1} \begin{Bmatrix} \lambda_1 & s_0 & s_3 \\ t_0 & \bar{l} & t_1 \end{Bmatrix} \begin{Bmatrix} \lambda_i & \lambda_j & \lambda_1 \\ q_1 & q_2 & \bar{l} \\ l'_{12} & l_{12} & t_1 \end{Bmatrix} \begin{Bmatrix} l'_{12} & l_{12} & t_1 \\ S'_{12} & S_{12} & s_0 \\ J'_{12} & J_{12} & t_0 \end{Bmatrix} \\
&\times \sum_{t_4} \hat{t}_4^2 (-1)^{t_4} \begin{Bmatrix} s_4 & 1 & s_3 \\ t_0 & \bar{l} & t_4 \end{Bmatrix} \begin{Bmatrix} \lambda_{3k} & \lambda_{4l} & s_4 \\ q_3 & q_4 & \bar{l} \\ l'_3 & l'_3 & t_4 \end{Bmatrix} \begin{Bmatrix} l'_3 & l_3 & t_4 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j'_3 & j_3 & t_0 \end{Bmatrix} \delta_{l'_{12}t_2} \delta_{l_{12}t_3} \delta_{l'_3t_5} \delta_{l_3t_6} .
\end{aligned}
\tag{190}
$$

$$ {}_A\langle [[\bullet\bullet]\bullet]_{JT} | V_{3N} | [[\bullet\bullet]\bullet]_{JT} \rangle_A $$

1. Complexity

for each **configuration**:

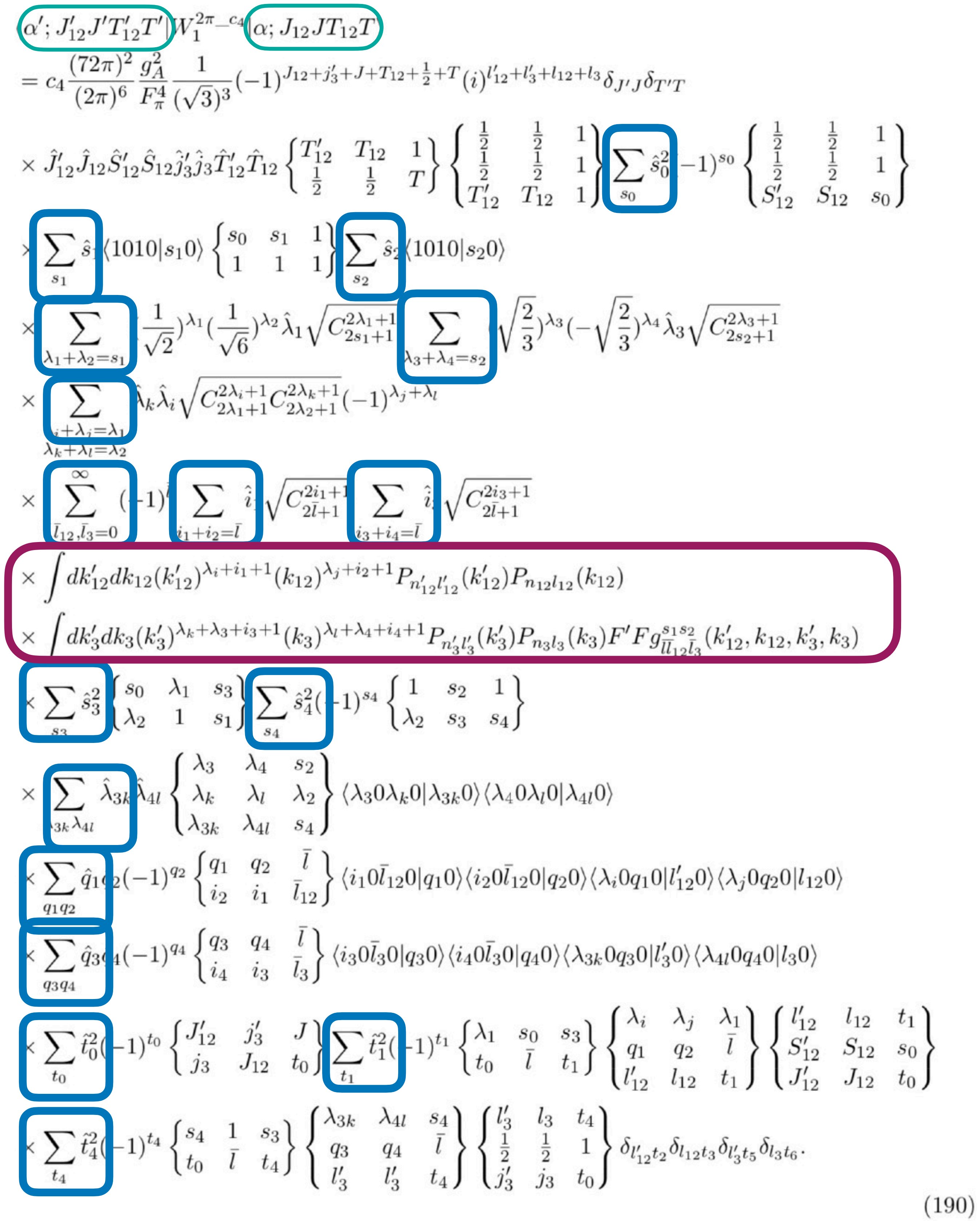
- **Summation**

  different range by configuration

- **Integration**

  High dimension (7)
  with 15 indices

Computation challenge



$$_A\langle [[\bullet\bullet]\bullet]_{JT} | V_{3N} | [[\bullet\bullet]\bullet]_{JT}\rangle_A$$

1. Complexity

for each **configuration**:

- **Summation**

  different range by configuration

- **Integration**

  High dimension (7)
  with 15 indices

2. Huge number of $\langle abc|V_{3N}|def\rangle$

$$a, b, \cdots \in H.O.(2n + l \leq E_{cut})$$

With $E_{cut}^{3N} = 12$

Needs **18Gb** around Memory for storage

General strategy

$$
\begin{aligned}
&\langle\alpha'; J'_{12}J'T'_{12}T'|W_1^{2\pi\_c_4}|\alpha; J_{12}JT_{12}T\rangle \\
&= c_4\frac{(72\pi)^2}{(2\pi)^6}\frac{g_A^2}{F_\pi^4}\frac{1}{(\sqrt{3})^3}(-1)^{J_{12}+j'_3+J+T_{12}+\frac{1}{2}+T}(i)^{l'_{12}+l'_3+l_{12}+l_3}\delta_{J'J}\delta_{T'T} \\
&\times \hat{J}'_{12}\hat{J}_{12}\hat{S}'_{12}\hat{S}_{12}\hat{j}'_3\hat{j}_3\hat{T}'_{12}\hat{T}_{12}\begin{Bmatrix} T'_{12} & T_{12} & 1 \\ \frac{1}{2} & \frac{1}{2} & T \end{Bmatrix}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ T'_{12} & T_{12} & 1 \end{Bmatrix}\sum_{s_0}\hat{s}_0^2(-1)^{s_0}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ S'_{12} & S_{12} & s_0 \end{Bmatrix} \\
&\times \sum_{s_1}\hat{s}_1\langle 1010|s_1 0\rangle\begin{Bmatrix} s_0 & s_1 & 1 \\ 1 & 1 & 1 \end{Bmatrix}\sum_{s_2}\hat{s}_2\langle 1010|s_2 0\rangle \\
&\times \sum_{\lambda_1+\lambda_2=s_1}(\frac{1}{\sqrt{2}})^{\lambda_1}(\frac{1}{\sqrt{6}})^{\lambda_2}\hat{\lambda}_1\sqrt{C_{2s_1+1}^{2\lambda_1+1}}\sum_{\lambda_3+\lambda_4=s_2}(\sqrt{\frac{2}{3}})^{\lambda_3}(-\sqrt{\frac{2}{3}})^{\lambda_4}\hat{\lambda}_3\sqrt{C_{2s_2+1}^{2\lambda_3+1}} \\
&\times \sum_{\substack{\lambda_i+\lambda_j=\lambda_1 \\ \lambda_k+\lambda_l=\lambda_2}}\hat{\lambda}_k\hat{\lambda}_i\sqrt{C_{2\lambda_1+1}^{2\lambda_i+1}C_{2\lambda_2+1}^{2\lambda_k+1}}(-1)^{\lambda_j+\lambda_l} \\
&\times \sum_{\bar{l},\bar{l}_{12},\bar{l}_3=0}^{\infty}(-1)^{\bar{l}}\sum_{i_1+i_2=\bar{l}}\hat{i}_1\sqrt{C_{2\bar{l}+1}^{2i_1+1}}\sum_{i_3+i_4=\bar{l}}\hat{i}_3\sqrt{C_{2\bar{l}+1}^{2i_3+1}} \\
&\times \int dk'_{12}dk_{12}(k'_{12})^{\lambda_i+i_1+1}(k_{12})^{\lambda_j+i_2+1}P_{n'_{12}l'_{12}}(k'_{12})P_{n_{12}l_{12}}(k_{12}) \\
&\times \int dk'_3 dk_3(k'_3)^{\lambda_k+\lambda_3+i_3+1}(k_3)^{\lambda_l+\lambda_4+i_4+1}P_{n'_3l'_3}(k'_3)P_{n_3l_3}(k_3)F'Fg_{\bar{l}\bar{l}_{12}\bar{l}_3}^{s_1s_2}(k'_{12},k_{12},k'_3,k_3) \\
&\times \sum_{s_3}\hat{s}_3^2\begin{Bmatrix} s_0 & \lambda_1 & s_3 \\ \lambda_2 & 1 & s_1 \end{Bmatrix}\sum_{s_4}\hat{s}_4^2(-1)^{s_4}\begin{Bmatrix} 1 & s_2 & 1 \\ \lambda_2 & s_3 & s_4 \end{Bmatrix} \\
&\times \sum_{\lambda_{3k}\lambda_{4l}}\hat{\lambda}_{3k}\hat{\lambda}_{4l}\begin{Bmatrix} \lambda_3 & \lambda_4 & s_2 \\ \lambda_k & \lambda_l & \lambda_2 \\ \lambda_{3k} & \lambda_{4l} & s_4 \end{Bmatrix}\langle\lambda_3 0\lambda_k 0|\lambda_{3k}0\rangle\langle\lambda_4 0\lambda_l 0|\lambda_{4l}0\rangle \\
&\times \sum_{q_1q_2}\hat{q}_1\hat{q}_2(-1)^{q_2}\begin{Bmatrix} q_1 & q_2 & \bar{l} \\ i_2 & i_1 & \bar{l}_{12} \end{Bmatrix}\langle i_1 0\bar{l}_{12}0|q_1 0\rangle\langle i_2 0\bar{l}_{12}0|q_2 0\rangle\langle\lambda_i 0q_1 0|l'_{12}0\rangle\langle\lambda_j 0q_2 0|l_{12}0\rangle \\
&\times \sum_{q_3q_4}\hat{q}_3\hat{q}_4(-1)^{q_4}\begin{Bmatrix} q_3 & q_4 & \bar{l} \\ i_4 & i_3 & \bar{l}_3 \end{Bmatrix}\langle i_3 0\bar{l}_3 0|q_3 0\rangle\langle i_4 0\bar{l}_3 0|q_4 0\rangle\langle\lambda_{3k}0q_3 0|l'_3 0\rangle\langle\lambda_{4l}0q_4 0|l_3 0\rangle \\
&\times \sum_{t_0}\hat{t}_0^2(-1)^{t_0}\begin{Bmatrix} J'_{12} & j'_3 & J \\ j_3 & J_{12} & t_0 \end{Bmatrix}\sum_{t_1}\hat{t}_1^2(-1)^{t_1}\begin{Bmatrix} \lambda_1 & s_0 & s_3 \\ t_0 & \bar{l} & t_1 \end{Bmatrix}\begin{Bmatrix} \lambda_i & \lambda_j & \lambda_1 \\ q_1 & q_2 & \bar{l} \\ l'_{12} & l_{12} & t_1 \end{Bmatrix}\begin{Bmatrix} l'_{12} & l_{12} & t_1 \\ S'_{12} & S_{12} & s_0 \\ J'_{12} & J_{12} & t_0 \end{Bmatrix} \\
&\times \sum_{t_4}\hat{t}_4^2(-1)^{t_4}\begin{Bmatrix} s_4 & 1 & s_3 \\ t_0 & \bar{l} & t_4 \end{Bmatrix}\begin{Bmatrix} \lambda_{3k} & \lambda_{4l} & s_4 \\ q_3 & q_4 & \bar{l} \\ l'_3 & l'_3 & t_4 \end{Bmatrix}\begin{Bmatrix} l'_3 & l_3 & t_4 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j'_3 & j_3 & t_0 \end{Bmatrix}\delta_{l'_{12}t_2}\delta_{l_{12}t_3}\delta_{l'_3t_5}\delta_{l_3t_6}.
\end{aligned}
\tag{190}
$$

More efficiency

- **Reduce redundant configuration**

  serialize the summation
  find&delete configuration with zero coefficients

General strategy

$$
\begin{aligned}
&\langle \alpha'; J'_{12}J'T'_{12}T'|W_1^{2\pi\_c_4}|\alpha; J_{12}JT_{12}T\rangle \\
&= c_4 \frac{(72\pi)^2}{(2\pi)^6}\frac{g_A^2}{F_\pi^4}\frac{1}{(\sqrt{3})^3}(-1)^{J_{12}+j'_3+J+T_{12}+\frac{1}{2}+T}(i)^{l'_{12}+l'_3+l_{12}+l_3}\delta_{J'J}\delta_{T'T} \\
&\times \hat{J}'_{12}\hat{J}_{12}\hat{S}'_{12}\hat{S}_{12}\hat{j}'_3\hat{j}_3\hat{T}'_{12}\hat{T}_{12}\begin{Bmatrix} T'_{12} & T_{12} & 1 \\ \frac{1}{2} & \frac{1}{2} & T \end{Bmatrix}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ T'_{12} & T_{12} & 1 \end{Bmatrix}\sum_{s_0}\hat{s}_0^2(-1)^{s_0}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ S'_{12} & S_{12} & s_0 \end{Bmatrix} \\
&\times \sum_{s_1}\hat{s}_1\langle 1010|s_1 0\rangle \begin{Bmatrix} s_0 & s_1 & 1 \\ 1 & 1 & 1 \end{Bmatrix}\sum_{s_2}\hat{s}_2\langle 1010|s_2 0\rangle \\
&\times \sum_{\lambda_1+\lambda_2=s_1}(\frac{1}{\sqrt{2}})^{\lambda_1}(\frac{1}{\sqrt{6}})^{\lambda_2}\hat{\lambda}_1\sqrt{C_{2s_1+1}^{2\lambda_1+1}}\sum_{\lambda_3+\lambda_4=s_2}(\sqrt{\frac{2}{3}})^{\lambda_3}(-\sqrt{\frac{2}{3}})^{\lambda_4}\hat{\lambda}_3\sqrt{C_{2s_2+1}^{2\lambda_3+1}} \\
&\times \sum_{\substack{\lambda_i+\lambda_j=\lambda_1 \\ \lambda_k+\lambda_l=\lambda_2}}\hat{\lambda}_k\hat{\lambda}_i\sqrt{C_{2\lambda_1+1}^{2\lambda_i+1}C_{2\lambda_2+1}^{2\lambda_k+1}}(-1)^{\lambda_j+\lambda_l} \\
&\times \sum_{\bar{l},\bar{l}_{12},\bar{l}_3=0}^{\infty}(-1)^{\bar{l}}\sum_{i_1+i_2=\bar{l}}\hat{i}_1\sqrt{C_{2\bar{l}+1}^{2i_1+1}}\sum_{i_3+i_4=\bar{l}}\hat{i}_3\sqrt{C_{2\bar{l}+1}^{2i_3+1}} \\
&\times \int dk'_{12}dk_{12}(k'_{12})^{\lambda_i+i_1+1}(k_{12})^{\lambda_j+i_2+1}P_{n'_{12}l'_{12}}(k'_{12})P_{n_{12}l_{12}}(k_{12}) \\
&\times \int dk'_3 dk_3(k'_3)^{\lambda_k+\lambda_3+i_3+1}(k_3)^{\lambda_l+\lambda_4+i_4+1}P_{n'_3l'_3}(k'_3)P_{n_3l_3}(k_3)F'Fg_{\bar{l}\bar{l}_{12}\bar{l}_3}^{s_1s_2}(k'_{12},k_{12},k'_3,k_3) \\
&\times \sum_{s_3}\hat{s}_3^2\begin{Bmatrix} s_0 & \lambda_1 & s_3 \\ \lambda_2 & 1 & s_1 \end{Bmatrix}\sum_{s_4}\hat{s}_4^2(-1)^{s_4}\begin{Bmatrix} 1 & s_2 & 1 \\ \lambda_2 & s_3 & s_4 \end{Bmatrix} \\
&\times \sum_{\lambda_{3k}\lambda_{4l}}\hat{\lambda}_{3k}\hat{\lambda}_{4l}\begin{Bmatrix} \lambda_3 & \lambda_4 & s_2 \\ \lambda_k & \lambda_l & \lambda_2 \\ \lambda_{3k} & \lambda_{4l} & s_4 \end{Bmatrix}\langle\lambda_3 0\lambda_k 0|\lambda_{3k}0\rangle\langle\lambda_4 0\lambda_l 0|\lambda_{4l}0\rangle \\
&\times \sum_{q_1q_2}\hat{q}_1\hat{q}_2(-1)^{q_2}\begin{Bmatrix} q_1 & q_2 & \bar{l} \\ i_2 & i_1 & \bar{l}_{12} \end{Bmatrix}\langle i_1 0\bar{l}_{12}0|q_1 0\rangle\langle i_2 0\bar{l}_{12}0|q_2 0\rangle\langle\lambda_i 0 q_1 0|l'_{12}0\rangle\langle\lambda_j 0 q_2 0|l_{12}0\rangle \\
&\times \sum_{q_3q_4}\hat{q}_3\hat{q}_4(-1)^{q_4}\begin{Bmatrix} q_3 & q_4 & \bar{l} \\ i_4 & i_3 & \bar{l}_3 \end{Bmatrix}\langle i_3 0\bar{l}_3 0|q_3 0\rangle\langle i_4 0\bar{l}_3 0|q_4 0\rangle\langle\lambda_{3k}0 q_3 0|l'_3 0\rangle\langle\lambda_{4l}0 q_4 0|l_3 0\rangle \\
&\times \sum_{t_0}\hat{t}_0^2(-1)^{t_0}\begin{Bmatrix} J'_{12} & j'_3 & J \\ j_3 & J_{12} & t_0 \end{Bmatrix}\sum_{t_1}\hat{t}_1^2(-1)^{t_1}\begin{Bmatrix} \lambda_1 & s_0 & s_3 \\ t_0 & \bar{l} & t_1 \end{Bmatrix}\begin{Bmatrix} \lambda_i & \lambda_j & \lambda_1 \\ q_1 & q_2 & \bar{l} \\ l'_{12} & l_{12} & t_1 \end{Bmatrix}\begin{Bmatrix} l'_{12} & l_{12} & t_1 \\ S'_{12} & S_{12} & s_0 \\ J'_{12} & J_{12} & t_0 \end{Bmatrix} \\
&\times \sum_{t_4}\hat{t}_4^2(-1)^{t_4}\begin{Bmatrix} s_4 & 1 & s_3 \\ t_0 & \bar{l} & t_4 \end{Bmatrix}\begin{Bmatrix} \lambda_{3k} & \lambda_{4l} & s_4 \\ q_3 & q_4 & \bar{l} \\ l'_3 & l'_3 & t_4 \end{Bmatrix}\begin{Bmatrix} l'_3 & l_3 & t_4 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j'_3 & j_3 & t_0 \end{Bmatrix}\delta_{l'_{12}t_2}\delta_{l_{12}t_3}\delta_{l'_3t_5}\delta_{l_3t_6}.
\end{aligned}
\tag{190}
$$

More efficiency

- **Reduce redundant configuration**

  serialize the summation
  find&delete configuration with zero coefficients

  Reduce the total number of integration

$$PFKglbarS_{nl}[\lambda_i, \lambda_j, \lambda_k, \lambda_l, \lambda_3, \lambda_4, i_1, i_2, i_3, i_4, \bar{l}, \bar{l}_{12}, \bar{l}_3, s_1, s_2]$$

General strategy

$$
\begin{aligned}
&\langle\alpha'; J'_{12}J'T'_{12}T'|W_1^{2\pi\_c_4}|\alpha; J_{12}JT_{12}T\rangle \\
&= c_4\frac{(72\pi)^2}{(2\pi)^6}\frac{g_A^2}{F_\pi^4}\frac{1}{(\sqrt{3})^3}(-1)^{J_{12}+j'_3+J+T_{12}+\frac{1}{2}+T}(i)^{l'_{12}+l'_3+l_{12}+l_3}\delta_{J'J}\delta_{T'T} \\
&\times \hat{J}'_{12}\hat{J}_{12}\hat{S}'_{12}\hat{S}_{12}\hat{j}'_3\hat{j}_3\hat{T}'_{12}\hat{T}_{12}\begin{Bmatrix}T'_{12} & T_{12} & 1\\ \frac{1}{2} & \frac{1}{2} & T\end{Bmatrix}\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ \frac{1}{2} & \frac{1}{2} & 1\\ T'_{12} & T_{12} & 1\end{Bmatrix}\sum_{s_0}\hat{s}_0^2(-1)^{s_0}\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ \frac{1}{2} & \frac{1}{2} & 1\\ S'_{12} & S_{12} & s_0\end{Bmatrix} \\
&\times \sum_{s_1}\hat{s}_1\langle 1010|s_1 0\rangle\begin{Bmatrix}s_0 & s_1 & 1\\ 1 & 1 & 1\end{Bmatrix}\sum_{s_2}\hat{s}_2\langle 1010|s_2 0\rangle \\
&\times \sum_{\lambda_1+\lambda_2=s_1}(\frac{1}{\sqrt{2}})^{\lambda_1}(\frac{1}{\sqrt{6}})^{\lambda_2}\hat{\lambda}_1\sqrt{C_{2s_1+1}^{2\lambda_1+1}}\sum_{\lambda_3+\lambda_4=s_2}(\sqrt{\frac{2}{3}})^{\lambda_3}(-\sqrt{\frac{2}{3}})^{\lambda_4}\hat{\lambda}_3\sqrt{C_{2s_2+1}^{2\lambda_3+1}} \\
&\times \sum_{\substack{\lambda_i+\lambda_j=\lambda_1\\ \lambda_k+\lambda_l=\lambda_2}}\hat{\lambda}_k\hat{\lambda}_i\sqrt{C_{2\lambda_1+1}^{2\lambda_i+1}C_{2\lambda_2+1}^{2\lambda_k+1}}(-1)^{\lambda_j+\lambda_l} \\
&\times \sum_{\bar{l},\bar{l}_{12},\bar{l}_3=0}^{\infty}(-1)^{\bar{l}}\sum_{i_1+i_2=\bar{l}}\hat{i}_1\sqrt{C_{2\bar{l}+1}^{2i_1+1}}\sum_{i_3+i_4=\bar{l}}\hat{i}_3\sqrt{C_{2\bar{l}+1}^{2i_3+1}} \\
&\times \int dk'_{12}dk_{12}(k'_{12})^{\lambda_i+i_1+1}(k_{12})^{\lambda_j+i_2+1}P_{n'_{12}l'_{12}}(k'_{12})P_{n_{12}l_{12}}(k_{12}) \\
&\times \int dk'_3dk_3(k'_3)^{\lambda_k+\lambda_3+i_3+1}(k_3)^{\lambda_l+\lambda_4+i_4+1}P_{n'_3l'_3}(k'_3)P_{n_3l_3}(k_3)F'Fg_{\bar{l}\bar{l}_{12}\bar{l}_3}^{s_1s_2}(k'_{12},k_{12},k'_3,k_3) \\
&\times \sum_{s_3}\hat{s}_3^2\begin{Bmatrix}s_0 & \lambda_1 & s_3\\ \lambda_2 & 1 & s_1\end{Bmatrix}\sum_{s_4}\hat{s}_4^2(-1)^{s_4}\begin{Bmatrix}1 & s_2 & 1\\ \lambda_2 & s_3 & s_4\end{Bmatrix} \\
&\times \sum_{\lambda_{3k}\lambda_{4l}}\hat{\lambda}_{3k}\hat{\lambda}_{4l}\begin{Bmatrix}\lambda_3 & \lambda_4 & s_2\\ \lambda_k & \lambda_l & \lambda_2\\ \lambda_{3k} & \lambda_{4l} & s_4\end{Bmatrix}\langle\lambda_3 0\lambda_k 0|\lambda_{3k}0\rangle\langle\lambda_4 0\lambda_l 0|\lambda_{4l}0\rangle \\
&\times \sum_{q_1q_2}\hat{q}_1\hat{q}_2(-1)^{q_2}\begin{Bmatrix}q_1 & q_2 & \bar{l}\\ i_2 & i_1 & \bar{l}_{12}\end{Bmatrix}\langle i_1 0\bar{l}_{12}0|q_1 0\rangle\langle i_2 0\bar{l}_{12}0|q_2 0\rangle\langle\lambda_i 0q_1 0|l'_{12}0\rangle\langle\lambda_j 0q_2 0|l_{12}0\rangle \\
&\times \sum_{q_3q_4}\hat{q}_3\hat{q}_4(-1)^{q_4}\begin{Bmatrix}q_3 & q_4 & \bar{l}\\ i_4 & i_3 & \bar{l}_3\end{Bmatrix}\langle i_3 0\bar{l}_3 0|q_3 0\rangle\langle i_4 0\bar{l}_3 0|q_4 0\rangle\langle\lambda_{3k}0q_3 0|l'_3 0\rangle\langle\lambda_{4l}0q_4 0|l_3 0\rangle \\
&\times \sum_{t_0}\hat{t}_0^2(-1)^{t_0}\begin{Bmatrix}J'_{12} & j'_3 & J\\ j_3 & J_{12} & t_0\end{Bmatrix}\sum_{t_1}\hat{t}_1^2(-1)^{t_1}\begin{Bmatrix}\lambda_1 & s_0 & s_3\\ t_0 & \bar{l} & t_1\end{Bmatrix}\begin{Bmatrix}\lambda_i & \lambda_j & \lambda_1\\ q_1 & q_2 & \bar{l}\\ l'_{12} & l_{12} & t_1\end{Bmatrix}\begin{Bmatrix}l'_{12} & l_{12} & t_1\\ S'_{12} & S_{12} & s_0\\ J'_{12} & J_{12} & t_0\end{Bmatrix} \\
&\times \sum_{t_4}\hat{t}_4^2(-1)^{t_4}\begin{Bmatrix}s_4 & 1 & s_3\\ t_0 & \bar{l} & t_4\end{Bmatrix}\begin{Bmatrix}\lambda_{3k} & \lambda_{4l} & s_4\\ q_3 & q_4 & \bar{l}\\ l'_3 & l'_3 & t_4\end{Bmatrix}\begin{Bmatrix}l'_3 & l_3 & t_4\\ \frac{1}{2} & \frac{1}{2} & 1\\ j'_3 & j_3 & t_0\end{Bmatrix}\delta_{l'_{12}t_2}\delta_{l_{12}t_3}\delta_{l'_3t_5}\delta_{l_3t_6}.
\end{aligned}
\tag{190}
$$

More efficiency

- **Reduce redundant configuration**

  serialize the summation
  find&delete configuration with zero coefficients

  Reduce the total number of integration

  $PFKglbarS_{nl}[\lambda_i, \lambda_j, \lambda_k, \lambda_l, \lambda_3, \lambda_4, i_1, i_2, i_3, i_4, \bar{l}, \bar{l}_{12}, \bar{l}_3, s_1, s_2]$

- **Reduce redundant computation**

General strategy

$$
\begin{aligned}
&\langle \alpha'; J'_{12}J'T'_{12}T'|W_1^{2\pi-c_4}|\alpha; J_{12}JT_{12}T\rangle \\
&= c_4 \frac{(72\pi)^2}{(2\pi)^6}\frac{g_A^2}{F_\pi^4}\frac{1}{(\sqrt{3})^3}(-1)^{J_{12}+j'_3+J+T_{12}+\frac{1}{2}+T}(i)^{l'_{12}+l'_3+l_{12}+l_3}\delta_{J'J}\delta_{T'T} \\
&\times \hat{J}'_{12}\hat{J}_{12}\hat{S}'_{12}\hat{S}_{12}\hat{j}'_3\hat{j}_3\hat{T}'_{12}\hat{T}_{12}\begin{Bmatrix} T'_{12} & T_{12} & 1 \\ \frac{1}{2} & \frac{1}{2} & T \end{Bmatrix}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ T'_{12} & T_{12} & 1 \end{Bmatrix}\sum_{s_0}\hat{s}_0^2(-1)^{s_0}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ S'_{12} & S_{12} & s_0 \end{Bmatrix} \\
&\times \sum_{s_1}\hat{s}_1\langle 1010|s_1 0\rangle\begin{Bmatrix} s_0 & s_1 & 1 \\ 1 & 1 & 1 \end{Bmatrix}\sum_{s_2}\hat{s}_2\langle 1010|s_2 0\rangle \\
&\times \sum_{\lambda_1+\lambda_2=s_1}\left(\frac{1}{\sqrt{2}}\right)^{\lambda_1}\left(\frac{1}{\sqrt{6}}\right)^{\lambda_2}\hat{\lambda}_1\sqrt{C_{2s_1+1}^{2\lambda_1+1}}\sum_{\lambda_3+\lambda_4=s_2}\left(\sqrt{\frac{2}{3}}\right)^{\lambda_3}\left(-\sqrt{\frac{2}{3}}\right)^{\lambda_4}\hat{\lambda}_3\sqrt{C_{2s_2+1}^{2\lambda_3+1}} \\
&\times \sum_{\substack{\lambda_i+\lambda_j=\lambda_1 \\ \lambda_k+\lambda_l=\lambda_2}}\hat{\lambda}_k\hat{\lambda}_i\sqrt{C_{2\lambda_1+1}^{2\lambda_i+1}C_{2\lambda_2+1}^{2\lambda_k+1}}(-1)^{\lambda_j+\lambda_l} \\
&\times \sum_{\bar{l},\bar{l}_{12},\bar{l}_3=0}^{\infty}(-1)^{\bar{l}}\sum_{i_1+i_2=\bar{l}}\hat{i}_1\sqrt{C_{2\bar{l}+1}^{2i_1+1}}\sum_{i_3+i_4=\bar{l}}\hat{i}_3\sqrt{C_{2\bar{l}+1}^{2i_3+1}} \\
&\times \int dk'_{12}dk_{12}(k'_{12})^{\lambda_i+i_1+1}(k_{12})^{\lambda_j+i_2+1}P_{n'_{12}l'_{12}}(k'_{12})P_{n_{12}l_{12}}(k_{12}) \\
&\times \int dk'_3 dk_3(k'_3)^{\lambda_k+\lambda_3+i_3+1}(k_3)^{\lambda_l+\lambda_4+i_4+1}P_{n'_3l'_3}(k'_3)P_{n_3l_3}(k_3)F'Fg_{\bar{l}\bar{l}_{12}\bar{l}_3}^{s_1s_2}(k'_{12},k_{12},k'_3,k_3) \\
&\times \sum_{s_3}\hat{s}_3^2\begin{Bmatrix} s_0 & \lambda_1 & s_3 \\ \lambda_2 & 1 & s_1 \end{Bmatrix}\sum_{s_4}\hat{s}_4^2(-1)^{s_4}\begin{Bmatrix} 1 & s_2 & 1 \\ \lambda_2 & s_3 & s_4 \end{Bmatrix} \\
&\times \sum_{\lambda_{3k}\lambda_{4l}}\hat{\lambda}_{3k}\hat{\lambda}_{4l}\begin{Bmatrix} \lambda_3 & \lambda_4 & s_2 \\ \lambda_k & \lambda_l & \lambda_2 \\ \lambda_{3k} & \lambda_{4l} & s_4 \end{Bmatrix}\langle\lambda_3 0\lambda_k 0|\lambda_{3k}0\rangle\langle\lambda_4 0\lambda_l 0|\lambda_{4l}0\rangle \\
&\times \sum_{q_1q_2}\hat{q}_1\hat{q}_2(-1)^{q_2}\begin{Bmatrix} q_1 & q_2 & \bar{l} \\ i_2 & i_1 & \bar{l}_{12} \end{Bmatrix}\langle i_1 0\bar{l}_{12}0|q_1 0\rangle\langle i_2 0\bar{l}_{12}0|q_2 0\rangle\langle\lambda_i 0 q_1 0|l'_{12}0\rangle\langle\lambda_j 0 q_2 0|l_{12}0\rangle \\
&\times \sum_{q_3q_4}\hat{q}_3\hat{q}_4(-1)^{q_4}\begin{Bmatrix} q_3 & q_4 & \bar{l} \\ i_4 & i_3 & \bar{l}_3 \end{Bmatrix}\langle i_3 0\bar{l}_3 0|q_3 0\rangle\langle i_4 0\bar{l}_3 0|q_4 0\rangle\langle\lambda_{3k}0q_3 0|l'_3 0\rangle\langle\lambda_{4l}0q_4 0|l_3 0\rangle \\
&\times \sum_{t_0}\hat{t}_0^2(-1)^{t_0}\begin{Bmatrix} J'_{12} & j'_3 & J \\ j_3 & J_{12} & t_0 \end{Bmatrix}\sum_{t_1}\hat{t}_1^2(-1)^{t_1}\begin{Bmatrix} \lambda_1 & s_0 & s_3 \\ t_0 & \bar{l} & t_1 \end{Bmatrix}\begin{Bmatrix} \lambda_i & \lambda_j & \lambda_1 \\ q_1 & q_2 & \bar{l} \\ l'_{12} & l_{12} & t_1 \end{Bmatrix}\begin{Bmatrix} l'_{12} & l_{12} & t_1 \\ S'_{12} & S_{12} & s_0 \\ J'_{12} & J_{12} & t_0 \end{Bmatrix} \\
&\times \sum_{t_4}\hat{t}_4^2(-1)^{t_4}\begin{Bmatrix} s_4 & 1 & s_3 \\ t_0 & \bar{l} & t_4 \end{Bmatrix}\begin{Bmatrix} \lambda_{3k} & \lambda_{4l} & s_4 \\ q_3 & q_4 & \bar{l} \\ l'_3 & l'_3 & t_4 \end{Bmatrix}\begin{Bmatrix} l'_3 & l_3 & t_4 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j'_3 & j_3 & t_0 \end{Bmatrix}\delta_{l'_{12}t_2}\delta_{l_{12}t_3}\delta_{l'_3t_5}\delta_{l_3t_6}.
\end{aligned}
\tag{190}
$$

More efficiency

- **Reduce redundant configuration**

  serialize the summation
  find&delete configuration with zero coefficients

  Reduce the total number of integration

  $PFKglbarS_{nl}[\lambda_i,\lambda_j,\lambda_k,\lambda_l,\lambda_3,\lambda_4,i_1,i_2,i_3,i_4,\bar{l},\bar{l}_{12},\bar{l}_3,s_1,s_2]$

- **Reduce redundant computation**

  Calculate & Store the coefficients
  as much as possible

General strategy

$$
\begin{aligned}
&\langle \alpha'; J'_{12} J' T'_{12} T' | W_1^{2\pi\_c_4} | \alpha; J_{12} J T_{12} T \rangle \\
&= c_4 \frac{(72\pi)^2}{(2\pi)^6} \frac{g_A^2}{F_\pi^4} \frac{1}{(\sqrt{3})^3} (-1)^{J_{12}+j'_3+J+T_{12}+\frac{1}{2}+T} (i)^{l'_{12}+l'_3+l_{12}+l_3} \delta_{J'J} \delta_{T'T} \\
&\times \hat{J}'_{12} \hat{J}_{12} \hat{S}'_{12} \hat{S}_{12} \hat{j}'_3 \hat{j}_3 \hat{T}'_{12} \hat{T}_{12} \begin{Bmatrix} T'_{12} & T_{12} & 1 \\ \frac{1}{2} & \frac{1}{2} & T \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ T'_{12} & T_{12} & 1 \end{Bmatrix} \sum_{s_0} \hat{s}_0^2 (-1)^{s_0} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ S'_{12} & S_{12} & s_0 \end{Bmatrix} \\
&\times \sum_{s_1} \hat{s}_1 \langle 1010 | s_1 0 \rangle \begin{Bmatrix} s_0 & s_1 & 1 \\ 1 & 1 & 1 \end{Bmatrix} \sum_{s_2} \hat{s}_2 \langle 1010 | s_2 0 \rangle \\
&\times \sum_{\lambda_1+\lambda_2=s_1} \left(\frac{1}{\sqrt{2}}\right)^{\lambda_1} \left(\frac{1}{\sqrt{6}}\right)^{\lambda_2} \hat{\lambda}_1 \sqrt{C_{2s_1+1}^{2\lambda_1+1}} \sum_{\lambda_3+\lambda_4=s_2} \left(\sqrt{\frac{2}{3}}\right)^{\lambda_3} \left(-\sqrt{\frac{2}{3}}\right)^{\lambda_4} \hat{\lambda}_3 \sqrt{C_{2s_2+1}^{2\lambda_3+1}} \\
&\times \sum_{\substack{\lambda_i+\lambda_j=\lambda_1 \\ \lambda_k+\lambda_l=\lambda_2}} \hat{\lambda}_k \hat{\lambda}_i \sqrt{C_{2\lambda_1+1}^{2\lambda_i+1} C_{2\lambda_2+1}^{2\lambda_k+1}} (-1)^{\lambda_j+\lambda_l} \\
&\times \sum_{\bar{l},\bar{l}_{12},\bar{l}_3=0}^{\infty} (-1)^{\bar{l}} \sum_{i_1+i_2=\bar{l}} \hat{i}_1 \sqrt{C_{2\bar{l}+1}^{2i_1+1}} \sum_{i_3+i_4=\bar{l}} \hat{i}_3 \sqrt{C_{2\bar{l}+1}^{2i_3+1}} \\
&\times \int dk'_{12} dk_{12} (k'_{12})^{\lambda_i+i_1+1} (k_{12})^{\lambda_j+i_2+1} P_{n'_{12}l'_{12}}(k'_{12}) P_{n_{12}l_{12}}(k_{12}) \\
&\times \int dk'_3 dk_3 (k'_3)^{\lambda_k+\lambda_3+i_3+1} (k_3)^{\lambda_l+\lambda_4+i_4+1} P_{n'_3l'_3}(k'_3) P_{n_3l_3}(k_3) F' F g_{\bar{l}\bar{l}_{12}\bar{l}_3}^{s_1 s_2}(k'_{12}, k_{12}, k'_3, k_3) \\
&\times \sum_{s_3} \hat{s}_3^2 \begin{Bmatrix} s_0 & \lambda_1 & s_3 \\ \lambda_2 & 1 & s_1 \end{Bmatrix} \sum_{s_4} \hat{s}_4^2 (-1)^{s_4} \begin{Bmatrix} 1 & s_2 & 1 \\ \lambda_2 & s_3 & s_4 \end{Bmatrix} \\
&\times \sum_{\lambda_{3k}\lambda_{4l}} \hat{\lambda}_{3k} \hat{\lambda}_{4l} \begin{Bmatrix} \lambda_3 & \lambda_4 & s_2 \\ \lambda_k & \lambda_l & \lambda_2 \\ \lambda_{3k} & \lambda_{4l} & s_4 \end{Bmatrix} \langle \lambda_3 0 \lambda_k 0 | \lambda_{3k} 0 \rangle \langle \lambda_4 0 \lambda_l 0 | \lambda_{4l} 0 \rangle \\
&\times \sum_{q_1 q_2} \hat{q}_1 \hat{q}_2 (-1)^{q_2} \begin{Bmatrix} q_1 & q_2 & \bar{l} \\ i_2 & i_1 & \bar{l}_{12} \end{Bmatrix} \langle i_1 0 \bar{l}_{12} 0 | q_1 0 \rangle \langle i_2 0 \bar{l}_{12} 0 | q_2 0 \rangle \langle \lambda_i 0 q_1 0 | l'_{12} 0 \rangle \langle \lambda_j 0 q_2 0 | l_{12} 0 \rangle \\
&\times \sum_{q_3 q_4} \hat{q}_3 \hat{q}_4 (-1)^{q_4} \begin{Bmatrix} q_3 & q_4 & \bar{l} \\ i_4 & i_3 & \bar{l}_3 \end{Bmatrix} \langle i_3 0 \bar{l}_3 0 | q_3 0 \rangle \langle i_4 0 \bar{l}_3 0 | q_4 0 \rangle \langle \lambda_{3k} 0 q_3 0 | l'_3 0 \rangle \langle \lambda_{4l} 0 q_4 0 | l_3 0 \rangle \\
&\times \sum_{t_0} \hat{t}_0^2 (-1)^{t_0} \begin{Bmatrix} J'_{12} & j'_3 & J \\ j_3 & J_{12} & t_0 \end{Bmatrix} \sum_{t_1} \hat{t}_1^2 (-1)^{t_1} \begin{Bmatrix} \lambda_1 & s_0 & s_3 \\ t_0 & \bar{l} & t_1 \end{Bmatrix} \begin{Bmatrix} \lambda_i & \lambda_j & \lambda_1 \\ q_1 & q_2 & \bar{l} \\ l'_{12} & l_{12} & t_1 \end{Bmatrix} \begin{Bmatrix} l'_{12} & l_{12} & t_1 \\ S'_{12} & S_{12} & s_0 \\ J'_{12} & J_{12} & t_0 \end{Bmatrix} \\
&\times \sum_{t_4} \hat{t}_4^2 (-1)^{t_4} \begin{Bmatrix} s_4 & 1 & s_3 \\ t_0 & \bar{l} & t_4 \end{Bmatrix} \begin{Bmatrix} \lambda_{3k} & \lambda_{4l} & s_4 \\ q_3 & q_4 & \bar{l} \\ l'_3 & l'_3 & t_4 \end{Bmatrix} \begin{Bmatrix} l'_3 & l_3 & t_4 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j'_3 & j_3 & t_0 \end{Bmatrix} \delta_{l'_{12}t_2} \delta_{l_{12}t_3} \delta_{l'_3t_5} \delta_{l_3t_6} .
\end{aligned}
\tag{190}
$$

More efficiency

- **Reduce redundant configuration**

  serialize the summation
  find&delete configuration with zero coefficients

  Reduce the total number of integration

  $PFKglbarS_{nl}[\lambda_i, \lambda_j, \lambda_k, \lambda_l, \lambda_3, \lambda_4, i_1, i_2, i_3, i_4, \bar{l}, \bar{l}_{12}, \bar{l}_3, s_1, s_2]$

- **Reduce redundant computation**

  Calculate & Store the coefficients
  as much as possible

- **Others**

General strategy

$$
\begin{aligned}
&\langle\alpha'; J'_{12}J'T'_{12}T'|W_1^{2\pi\_c_4}|\alpha; J_{12}JT_{12}T\rangle \\
&= c_4\frac{(72\pi)^2}{(2\pi)^6}\frac{g_A^2}{F_\pi^4}\frac{1}{(\sqrt{3})^3}(-1)^{J_{12}+j'_3+J+T_{12}+\frac{1}{2}+T}(i)^{l'_{12}+l'_3+l_{12}+l_3}\delta_{J'J}\delta_{T'T} \\
&\times \hat{J}'_{12}\hat{J}_{12}\hat{S}'_{12}\hat{S}_{12}\hat{j}'_3\hat{j}_3\hat{T}'_{12}\hat{T}_{12}\begin{Bmatrix} T'_{12} & T_{12} & 1 \\ \frac{1}{2} & \frac{1}{2} & T \end{Bmatrix}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ T'_{12} & T_{12} & 1 \end{Bmatrix}\sum_{s_0}\hat{s}_0^2(-1)^{s_0}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ S'_{12} & S_{12} & s_0 \end{Bmatrix} \\
&\times \sum_{s_1}\hat{s}_1\langle 1010|s_1 0\rangle\begin{Bmatrix} s_0 & s_1 & 1 \\ 1 & 1 & 1 \end{Bmatrix}\sum_{s_2}\hat{s}_2\langle 1010|s_2 0\rangle \\
&\times \sum_{\lambda_1+\lambda_2=s_1}\left(\frac{1}{\sqrt{2}}\right)^{\lambda_1}\left(\frac{1}{\sqrt{6}}\right)^{\lambda_2}\hat{\lambda}_1\sqrt{C_{2s_1+1}^{2\lambda_1+1}}\sum_{\lambda_3+\lambda_4=s_2}\left(\sqrt{\frac{2}{3}}\right)^{\lambda_3}\left(-\sqrt{\frac{2}{3}}\right)^{\lambda_4}\hat{\lambda}_3\sqrt{C_{2s_2+1}^{2\lambda_3+1}} \\
&\times \sum_{\substack{\lambda_i+\lambda_j=\lambda_1 \\ \lambda_k+\lambda_l=\lambda_2}}\hat{\lambda}_k\hat{\lambda}_i\sqrt{C_{2\lambda_1+1}^{2\lambda_i+1}C_{2\lambda_2+1}^{2\lambda_k+1}}(-1)^{\lambda_j+\lambda_l} \\
&\times \sum_{\bar{l},\bar{l}_{12},\bar{l}_3=0}^{\infty}(-1)^{\bar{l}}\sum_{i_1+i_2=\bar{l}}\hat{i}_1\sqrt{C_{2\bar{l}+1}^{2i_1+1}}\sum_{i_3+i_4=\bar{l}}\hat{i}_3\sqrt{C_{2\bar{l}+1}^{2i_3+1}} \\
&\times \int dk'_{12}dk_{12}(k'_{12})^{\lambda_i+i_1+1}(k_{12})^{\lambda_j+i_2+1}P_{n'_{12}l'_{12}}(k'_{12})P_{n_{12}l_{12}}(k_{12}) \\
&\times \int dk'_3dk_3(k'_3)^{\lambda_k+\lambda_3+i_3+1}(k_3)^{\lambda_l+\lambda_4+i_4+1}P_{n'_3l'_3}(k'_3)P_{n_3l_3}(k_3)F'Fg^{s_1s_2}_{\bar{l}\bar{l}_{12}\bar{l}_3}(k'_{12},k_{12},k'_3,k_3) \\
&\times \sum_{s_3}\hat{s}_3^2\begin{Bmatrix} s_0 & \lambda_1 & s_3 \\ \lambda_2 & 1 & s_1 \end{Bmatrix}\sum_{s_4}\hat{s}_4^2(-1)^{s_4}\begin{Bmatrix} 1 & s_2 & 1 \\ \lambda_2 & s_3 & s_4 \end{Bmatrix} \\
&\times \sum_{\lambda_{3k}\lambda_{4l}}\hat{\lambda}_{3k}\hat{\lambda}_{4l}\begin{Bmatrix} \lambda_3 & \lambda_4 & s_2 \\ \lambda_k & \lambda_l & \lambda_2 \\ \lambda_{3k} & \lambda_{4l} & s_4 \end{Bmatrix}\langle\lambda_3 0\lambda_k 0|\lambda_{3k}0\rangle\langle\lambda_4 0\lambda_l 0|\lambda_{4l}0\rangle \\
&\times \sum_{q_1q_2}\hat{q}_1\hat{q}_2(-1)^{q_2}\begin{Bmatrix} q_1 & q_2 & \bar{l} \\ i_2 & i_1 & \bar{l}_{12} \end{Bmatrix}\langle i_1 0\bar{l}_{12}0|q_1 0\rangle\langle i_2 0\bar{l}_{12}0|q_2 0\rangle\langle\lambda_i 0 q_1 0|l'_{12}0\rangle\langle\lambda_j 0 q_2 0|l_{12}0\rangle \\
&\times \sum_{q_3q_4}\hat{q}_3\hat{q}_4(-1)^{q_4}\begin{Bmatrix} q_3 & q_4 & \bar{l} \\ i_4 & i_3 & \bar{l}_3 \end{Bmatrix}\langle i_3 0\bar{l}_3 0|q_3 0\rangle\langle i_4 0\bar{l}_3 0|q_4 0\rangle\langle\lambda_{3k}0 q_3 0|l'_3 0\rangle\langle\lambda_{4l}0 q_4 0|l_3 0\rangle \\
&\times \sum_{t_0}\hat{t}_0^2(-1)^{t_0}\begin{Bmatrix} J'_{12} & j'_3 & J \\ j_3 & J_{12} & t_0 \end{Bmatrix}\sum_{t_1}\hat{t}_1^2(-1)^{t_1}\begin{Bmatrix} \lambda_1 & s_0 & s_3 \\ t_0 & \bar{l} & t_1 \end{Bmatrix}\begin{Bmatrix} \lambda_i & \lambda_j & \lambda_1 \\ q_1 & q_2 & \bar{l} \\ l'_{12} & l_{12} & t_1 \end{Bmatrix}\begin{Bmatrix} l'_{12} & l_{12} & t_1 \\ S'_{12} & S_{12} & s_0 \\ J'_{12} & J_{12} & t_0 \end{Bmatrix} \\
&\times \sum_{t_4}\hat{t}_4^2(-1)^{t_4}\begin{Bmatrix} s_4 & 1 & s_3 \\ t_0 & \bar{l} & t_4 \end{Bmatrix}\begin{Bmatrix} \lambda_{3k} & \lambda_{4l} & s_4 \\ q_3 & q_4 & \bar{l} \\ l'_3 & l'_3 & t_4 \end{Bmatrix}\begin{Bmatrix} l'_3 & l_3 & t_4 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j'_3 & j_3 & t_0 \end{Bmatrix}\delta_{l'_{12}t_2}\delta_{l_{12}t_3}\delta_{l'_3t_5}\delta_{l_3t_6}.
\end{aligned}
\tag{190}
$$

More efficiency

- **Reduce redundant configuration**

  serialize the summation
  find&delete configuration with zero coefficients

  Reduce the total number of integration

  $PFKglbarS_{nl}[\lambda_i,\lambda_j,\lambda_k,\lambda_l,\lambda_3,\lambda_4,i_1,i_2,i_3,i_4,\bar{l},\bar{l}_{12},\bar{l}_3,s_1,s_2]$

- **Reduce redundant computation**

  Calculate & Store the coefficients
  as much as possible

- **Others** Sparse Matrix, Partical storage ……

General strategy

$$
\begin{aligned}
&\langle \alpha'; J'_{12}J'T'_{12}T'|W_1^{2\pi-c_4}|\alpha; J_{12}JT_{12}T\rangle \\
&= c_4 \frac{(72\pi)^2}{(2\pi)^6}\frac{g_A^2}{F_\pi^4}\frac{1}{(\sqrt{3})^3}(-1)^{J_{12}+j'_3+J+T_{12}+\frac{1}{2}+T}(i)^{l'_{12}+l'_3+l_{12}+l_3}\delta_{J'J}\delta_{T'T} \\
&\times \hat{J}'_{12}\hat{J}_{12}\hat{S}'_{12}\hat{S}_{12}\hat{j}'_3\hat{j}_3\hat{T}'_{12}\hat{T}_{12}\begin{Bmatrix} T'_{12} & T_{12} & 1 \\ \frac{1}{2} & \frac{1}{2} & T \end{Bmatrix}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ T'_{12} & T_{12} & 1 \end{Bmatrix}\sum_{s_0}\hat{s}_0^2(-1)^{s_0}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ S'_{12} & S_{12} & s_0 \end{Bmatrix} \\
&\times \sum_{s_1}\hat{s}_1\langle 1010|s_1 0\rangle\begin{Bmatrix} s_0 & s_1 & 1 \\ 1 & 1 & 1 \end{Bmatrix}\sum_{s_2}\hat{s}_2\langle 1010|s_2 0\rangle \\
&\times \sum_{\lambda_1+\lambda_2=s_1}\left(\frac{1}{\sqrt{2}}\right)^{\lambda_1}\left(\frac{1}{\sqrt{6}}\right)^{\lambda_2}\hat{\lambda}_1\sqrt{C_{2s_1+1}^{2\lambda_1+1}}\sum_{\lambda_3+\lambda_4=s_2}\left(\sqrt{\frac{2}{3}}\right)^{\lambda_3}\left(-\sqrt{\frac{2}{3}}\right)^{\lambda_4}\hat{\lambda}_3\sqrt{C_{2s_2+1}^{2\lambda_3+1}} \\
&\times \sum_{\substack{\lambda_i+\lambda_j=\lambda_1 \\ \lambda_k+\lambda_l=\lambda_2}}\hat{\lambda}_k\hat{\lambda}_i\sqrt{C_{2\lambda_1+1}^{2\lambda_i+1}C_{2\lambda_2+1}^{2\lambda_k+1}}(-1)^{\lambda_j+\lambda_l} \\
&\times \sum_{\bar{l},\bar{l}_{12},\bar{l}_3=0}^{\infty}(-1)^{\bar{l}}\sum_{i_1+i_2=\bar{l}}\hat{i}_1\sqrt{C_{2\bar{l}+1}^{2i_1+1}}\sum_{i_3+i_4=\bar{l}}\hat{i}_3\sqrt{C_{2\bar{l}+1}^{2i_3+1}} \\
&\times \int dk'_{12}dk_{12}(k'_{12})^{\lambda_i+i_1+1}(k_{12})^{\lambda_j+i_2+1}P_{n'_{12}l'_{12}}(k'_{12})P_{n_{12}l_{12}}(k_{12}) \\
&\times \int dk'_3 dk_3(k'_3)^{\lambda_k+\lambda_3+i_3+1}(k_3)^{\lambda_l+\lambda_4+i_4+1}P_{n'_3l'_3}(k'_3)P_{n_3l_3}(k_3)F'Fg_{\bar{l}\bar{l}_{12}\bar{l}_3}^{s_1s_2}(k'_{12},k_{12},k'_3,k_3) \\
&\times \sum_{s_3}\hat{s}_3^2\begin{Bmatrix} s_0 & \lambda_1 & s_3 \\ \lambda_2 & 1 & s_1 \end{Bmatrix}\sum_{s_4}\hat{s}_4^2(-1)^{s_4}\begin{Bmatrix} 1 & s_2 & 1 \\ \lambda_2 & s_3 & s_4 \end{Bmatrix} \\
&\times \sum_{\lambda_{3k}\lambda_{4l}}\hat{\lambda}_{3k}\hat{\lambda}_{4l}\begin{Bmatrix} \lambda_3 & \lambda_4 & s_2 \\ \lambda_k & \lambda_l & \lambda_2 \\ \lambda_{3k} & \lambda_{4l} & s_4 \end{Bmatrix}\langle\lambda_3 0\lambda_k 0|\lambda_{3k}0\rangle\langle\lambda_4 0\lambda_l 0|\lambda_{4l}0\rangle \\
&\times \sum_{q_1q_2}\hat{q}_1\hat{q}_2(-1)^{q_2}\begin{Bmatrix} q_1 & q_2 & \bar{l} \\ i_2 & i_1 & \bar{l}_{12} \end{Bmatrix}\langle i_1 0\bar{l}_{12}0|q_1 0\rangle\langle i_2 0\bar{l}_{12}0|q_2 0\rangle\langle\lambda_i 0 q_1 0|l'_{12}0\rangle\langle\lambda_j 0 q_2 0|l_{12}0\rangle \\
&\times \sum_{q_3q_4}\hat{q}_3\hat{q}_4(-1)^{q_4}\begin{Bmatrix} q_3 & q_4 & \bar{l} \\ i_4 & i_3 & \bar{l}_3 \end{Bmatrix}\langle i_3 0\bar{l}_3 0|q_3 0\rangle\langle i_4 0\bar{l}_3 0|q_4 0\rangle\langle\lambda_{3k}0 q_3 0|l'_3 0\rangle\langle\lambda_{4l}0 q_4 0|l_3 0\rangle \\
&\times \sum_{t_0}\hat{t}_0^2(-1)^{t_0}\begin{Bmatrix} J'_{12} & j'_3 & J \\ j_3 & J_{12} & t_0 \end{Bmatrix}\sum_{t_1}\hat{t}_1^2(-1)^{t_1}\begin{Bmatrix} \lambda_1 & s_0 & s_3 \\ t_0 & \bar{l} & t_1 \end{Bmatrix}\begin{Bmatrix} \lambda_i & \lambda_j & \lambda_1 \\ q_1 & q_2 & \bar{l} \\ l'_{12} & l_{12} & t_1 \end{Bmatrix}\begin{Bmatrix} l'_{12} & l_{12} & t_1 \\ S'_{12} & S_{12} & s_0 \\ J'_{12} & J_{12} & t_0 \end{Bmatrix} \\
&\times \sum_{t_4}\hat{t}_4^2(-1)^{t_4}\begin{Bmatrix} s_4 & 1 & s_3 \\ t_0 & \bar{l} & t_4 \end{Bmatrix}\begin{Bmatrix} \lambda_{3k} & \lambda_{4l} & s_4 \\ q_3 & q_4 & \bar{l} \\ l'_3 & l'_3 & t_4 \end{Bmatrix}\begin{Bmatrix} l'_3 & l_3 & t_4 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j'_3 & j_3 & t_0 \end{Bmatrix}\delta_{l'_{12}t_2}\delta_{l_{12}t_3}\delta_{l'_3t_5}\delta_{l_3t_6}.
\end{aligned} \tag{190}
$$

More efficiency

- **Reduce redundant configuration**

  serialize the summation
  find&delete configuration with zero coefficients

  Reduce the total number of integration

  $PFKglbarS_{nl}[\lambda_i, \lambda_j, \lambda_k, \lambda_l, \lambda_3, \lambda_4, i_1, i_2, i_3, i_4, \bar{l}, \bar{l}_{12}, \bar{l}_3, s_1, s_2]$

- **Reduce redundant computation**

  Calculate & Store the coefficients
  as much as possible

- **Others** Sparse Matrix, Partical storage ……

More Resources

- **Hybrid OpenMP&MPI**

  OpenMP - parallel by threads

  MPI - parallel by nodes

Computation Challenge

$${}_{as}\langle \tilde{a}'\tilde{b}'\tilde{c}'; J'_{ab}JT'_{ab}T|V_{3\mathrm{N}}|\tilde{a}\tilde{b}\tilde{c}; J_{ab}JT_{ab}T\rangle_{as} = 6 \sum_{\substack{N_{12},N_3,\alpha \\ N'_{12},N'_3,\alpha'}} \sum_{N_0L_0} \sum_{i,i'} \delta_{T_{ab}T_{12}} \delta_{T'_{ab}T'_{12}}$$

T coefficients: Jacobi -> (abc)JT state

$$\times\, T^{\tilde{a}'\tilde{b}'\tilde{c}'J'_{ab}J}_{N'_{12}N'_3\alpha'N_0L_0} T^{\tilde{a}\tilde{b}\tilde{c}J_{ab}J}_{N_{12}N_3\alpha N_0L_0}$$

M coefficients: Antisymmetry

$$\times\, M^{i'}_{N'_{12}N'_3\alpha'} M^{i}_{N_{12}N_3\alpha}$$

3BME in Jacobi basis

$$\times\, \langle N'i'J_{12,3}T, N_0L_0; J|V_{3\mathrm{N}}|NiJ_{12,3}T, N_0L_0; J\rangle$$

2018~2020: E3max <=8 (~300Mb)

2021~2022: E3max <=14 (~67Gb)

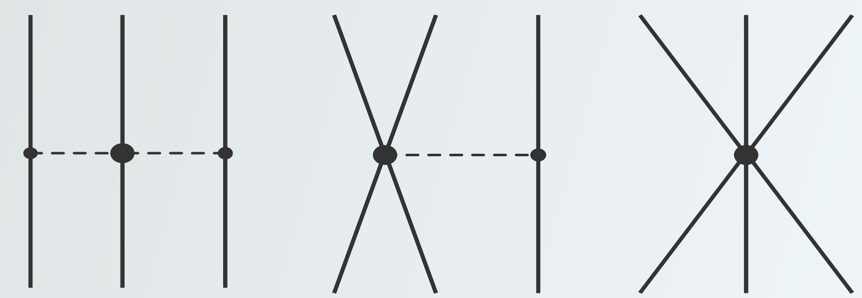
Worked by Shuang Zhang in PKU

- add a J_cut of Jacobi basis
- some parallel on T and M

$$V^{2\pi-exch}_{3N} = \frac{g_A^2}{8f_\pi^2} \sum_{i\neq j\neq k} \frac{\sigma_i \cdot \mathbf{Q}_i}{Q_i^2 + M_\pi^2} \frac{\sigma_j \cdot \mathbf{Q}'_j}{Q'^2_j + M_\pi^2} \times F^{\alpha\beta}_{ijk}\tau_i^\alpha\tau_j^\beta$$

$$F^{\alpha\beta}_{ijk}\tau_i^\alpha\tau_j^\beta = \delta_{\alpha\beta}[-4c_1m_\pi^2 + 2c_3\mathbf{q}_i\cdot\mathbf{q}_j] + c_4\epsilon_{\alpha\beta\gamma}\sigma_k\cdot(\mathbf{q}_i\times\mathbf{q}_j)$$

Calculation of TPE is complex



$V^{2\pi-exch}_{3N}$ $V^{1\pi+contact}_{3N}$ $V^{contact}_{3N}$

Contribution from 3NF

Hamiltonian with 3NF

$$H_{int} = (1 - \frac{1}{A})\sum_i \frac{p_i^2}{2m} + \sum_{i<j}(V_{ij}^{NN} - \frac{p_i \cdot p_j}{mA}) + \sum_{i<j<k} V_{ijk}^{3N}$$

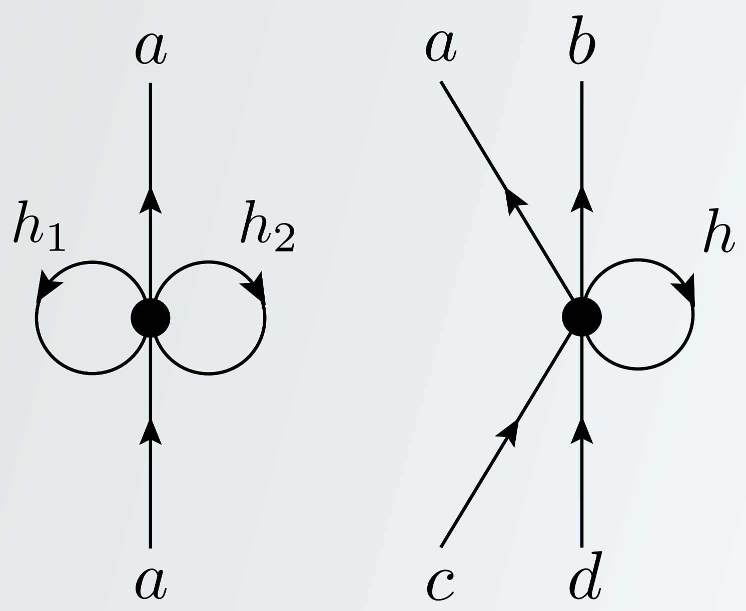
Contributions from three-body forces

1. one body

$$\langle j_a | 1b_{3N} | j_a \rangle = \sum_{\substack{h_1 h_2 \\ J_{12} J}} \frac{\hat{J}^2}{2\hat{j}_a^2} \langle [(j_{h_1} j_{h_2})_{J_{12}}, j_a]_J | V_{3N} | [(j_{h_1} j_{h_2})_{J_{12}}, j_a]_J \rangle$$

2. two body

$$\langle (j_a j_b)_J | 2b_{3N} | (j_c j_d)_J \rangle = \sum_{h J'} \frac{\hat{J}'^2}{\hat{J}^2} \langle [(j_a j_b)_J, j_h]_{J'} | V_{3N} | [(j_c j_d)_J, j_h]_{J'} \rangle$$



Normal ordering

Realistic Shell Model

A many-body Hamiltonian *H* defined in the full Hilbert space:

SP Num: 230 (j-scheme)
= 3542(m-scheme)

$$H = H_0 + H_1 = \sum_{i=1}^{A}(T_i + U_i) + \sum_{i<j}(V_{ij}^{NN} - U_i) + \sum_{i<j<k} V_{ijk}^{3N}$$

$C_{3000}^{8} \approx 1.6\times10^{23}$!

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Then, introducing a similarity transformation *X* :

$$\begin{pmatrix} PHP & PHQ \\ QHP & QHQ \end{pmatrix} \xrightarrow[\mathcal{QHP}=0]{\mathcal{H} = X^{-1}HX} \begin{pmatrix} P\mathcal{H}P & P\mathcal{H}Q \\ 0 & Q\mathcal{H}Q \end{pmatrix}$$

Suzuki & Lee : $X = e^{\omega}$ **with** $\omega = \begin{pmatrix} 0 & 0 \\ Q\omega P & 0 \end{pmatrix}$

$$H_1^{eff}(\omega) = PH_1P + PH_1Q\frac{1}{\epsilon - QHQ}QH_1P + PH_1Q\frac{1}{\epsilon - QHQ}\omega H_1{}^{eff}(\omega)$$

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$$Q\mathcal{H}P = 0$$

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Realistic Shell Model

This recursive equation for H_{eff} may be solved using iterative techniques (Krenciglowa-Kuo, Lee-Suzuki, Extended Krenciglowa-Kuo …)

$$H_{\mathrm{eff}} = \hat{Q} - \hat{Q}' \int \hat{Q} + \hat{Q}' \int \hat{Q} \int \hat{Q} - \hat{Q}' \int \hat{Q} \int \hat{Q} \int \hat{Q} \cdots$$

with $\hat{Q}$-box vertex function:

$$\hat{Q}(\epsilon) = PH_1P + PH_1Q \frac{1}{\epsilon - QHQ} QH_1P$$

For a many-body system, exact calculation of the $\hat{Q}$-box is prohibitive, then we perform a perturbative expansion:

L. Coraggio *et al.*, Annals of Physics 327 (2012)

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L. Coraggio *et al.*, Annals of Physics 327 (2012)

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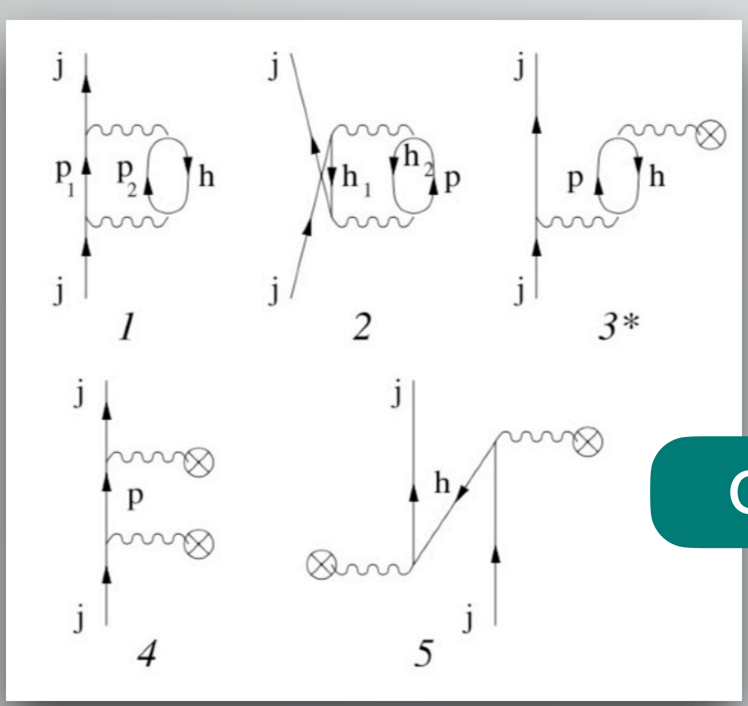
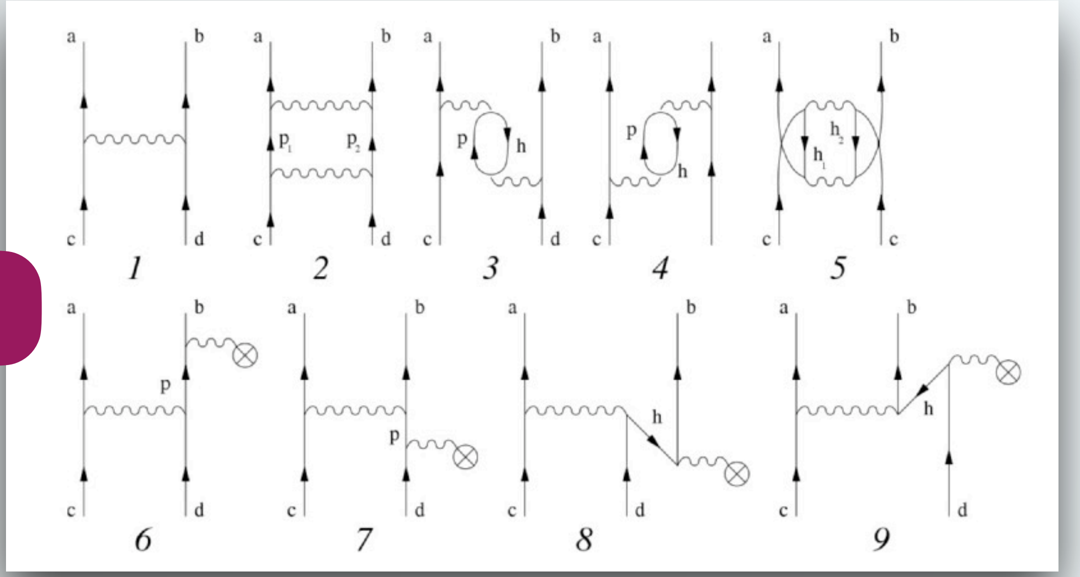


1st- & 2nd-ordrder

Two body

One body



3rd-ordrder: 126 diagrams

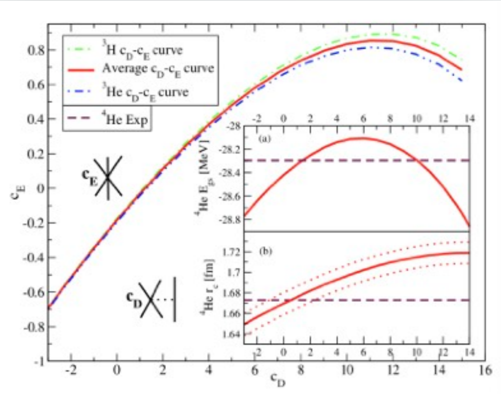
L. Coraggio *et al.*, Annals of Physics 327 (2012)

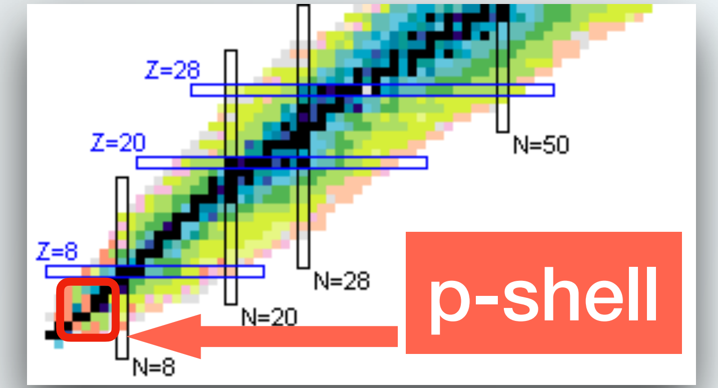
RSM for p-shell nuclei

Benchmark with NCSM

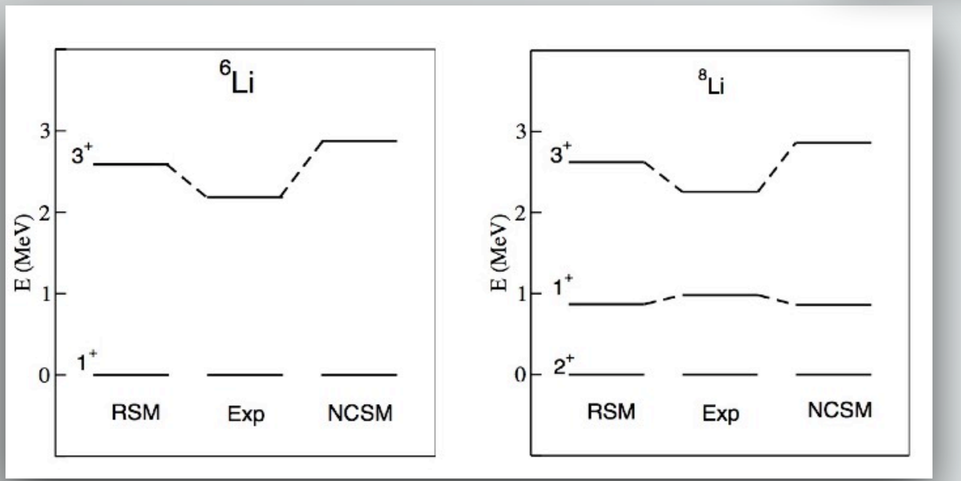
$$c_D = -1$$
$$c_E = -0.34$$

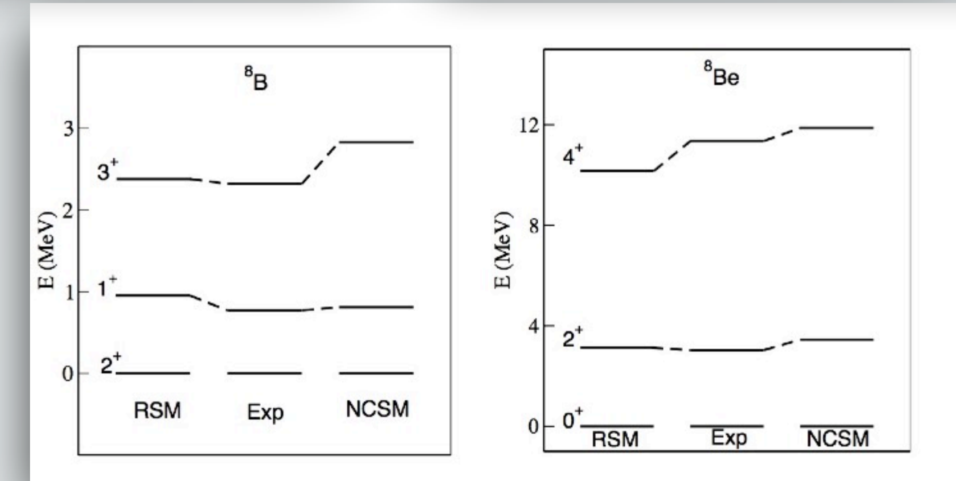
P. Navrátil, V. G. Gueorguiev,
J. P. Vary *et al.*, PRL **99**, 042501 (2007)



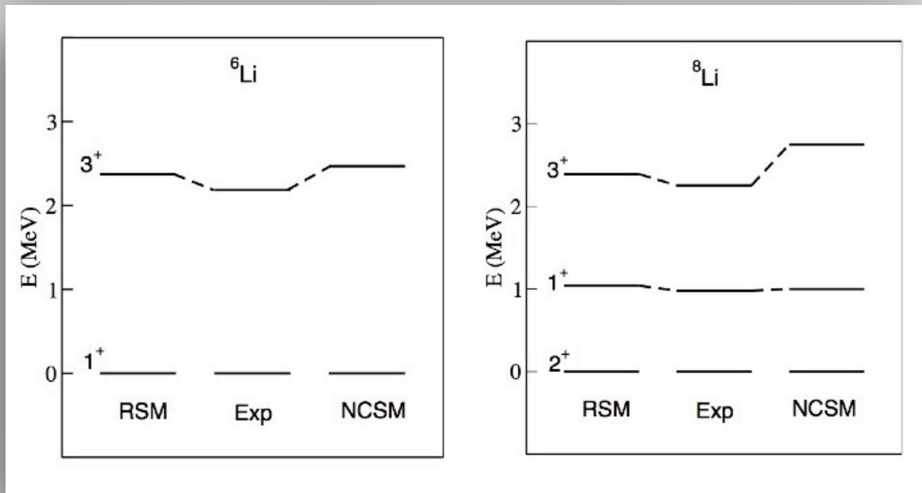


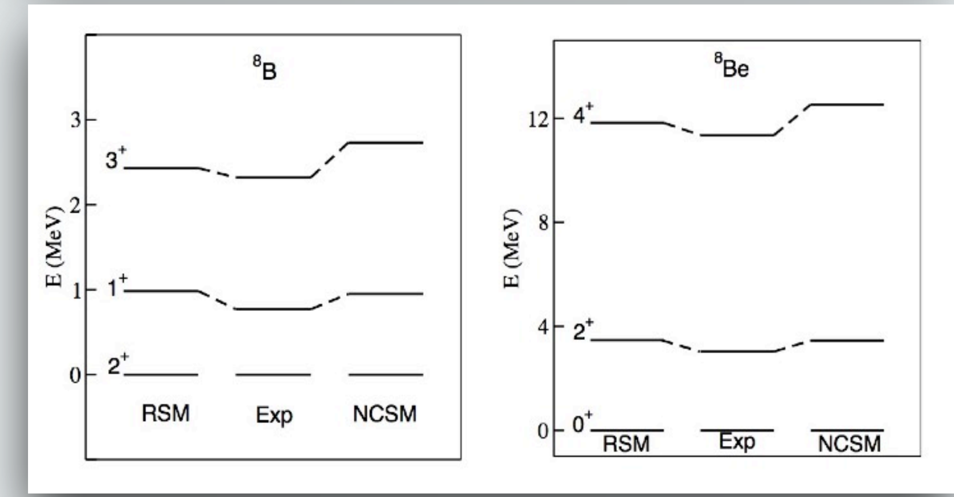
***NN* only**





NN* + *3N



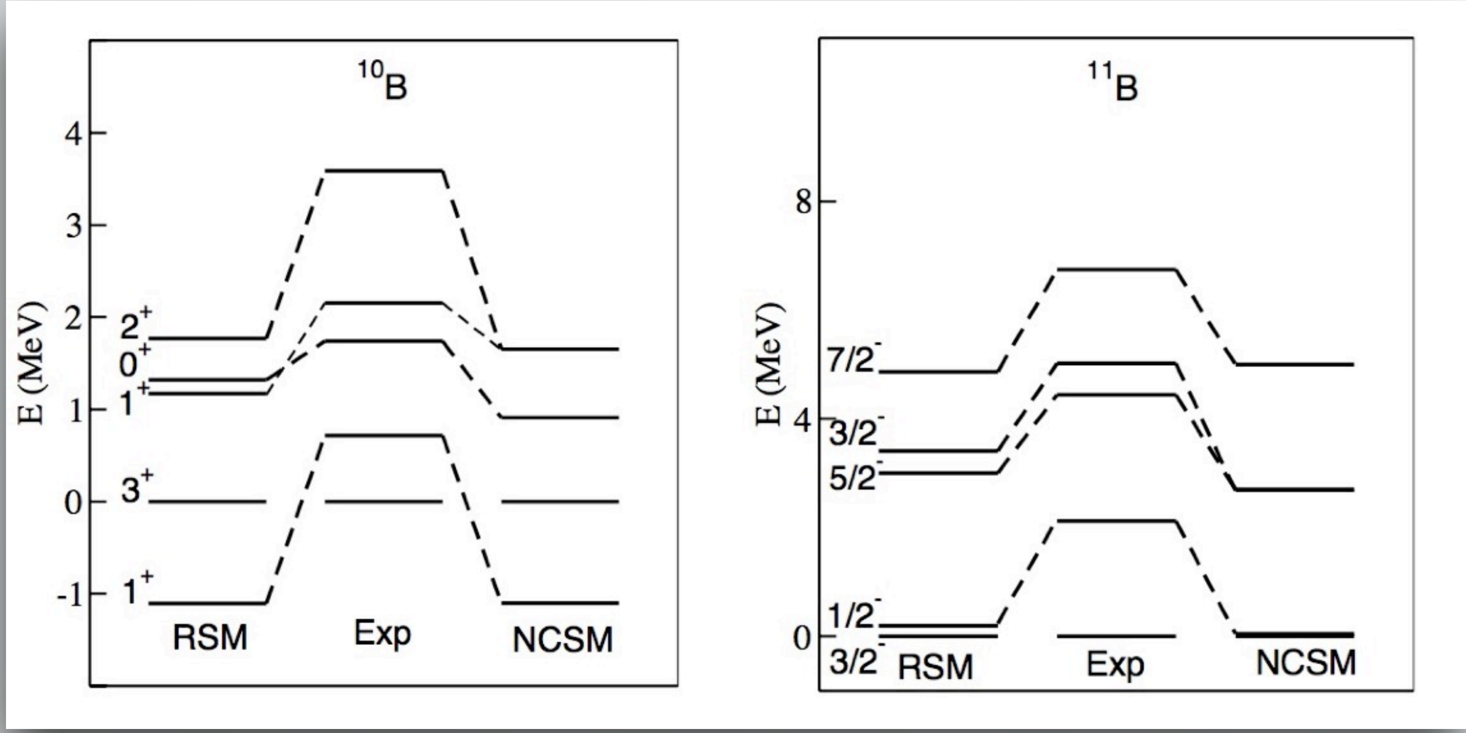


RSM: T. Fukui, L. De Angelis, **Y. Z. Ma** *et al.*, PRC **98**, 044305 (2018)
NCSM: J. P. Vary, P. Navratil, *et al.*, PRC 87, 014327 (2013); PRL **99,** 042501 (2007)

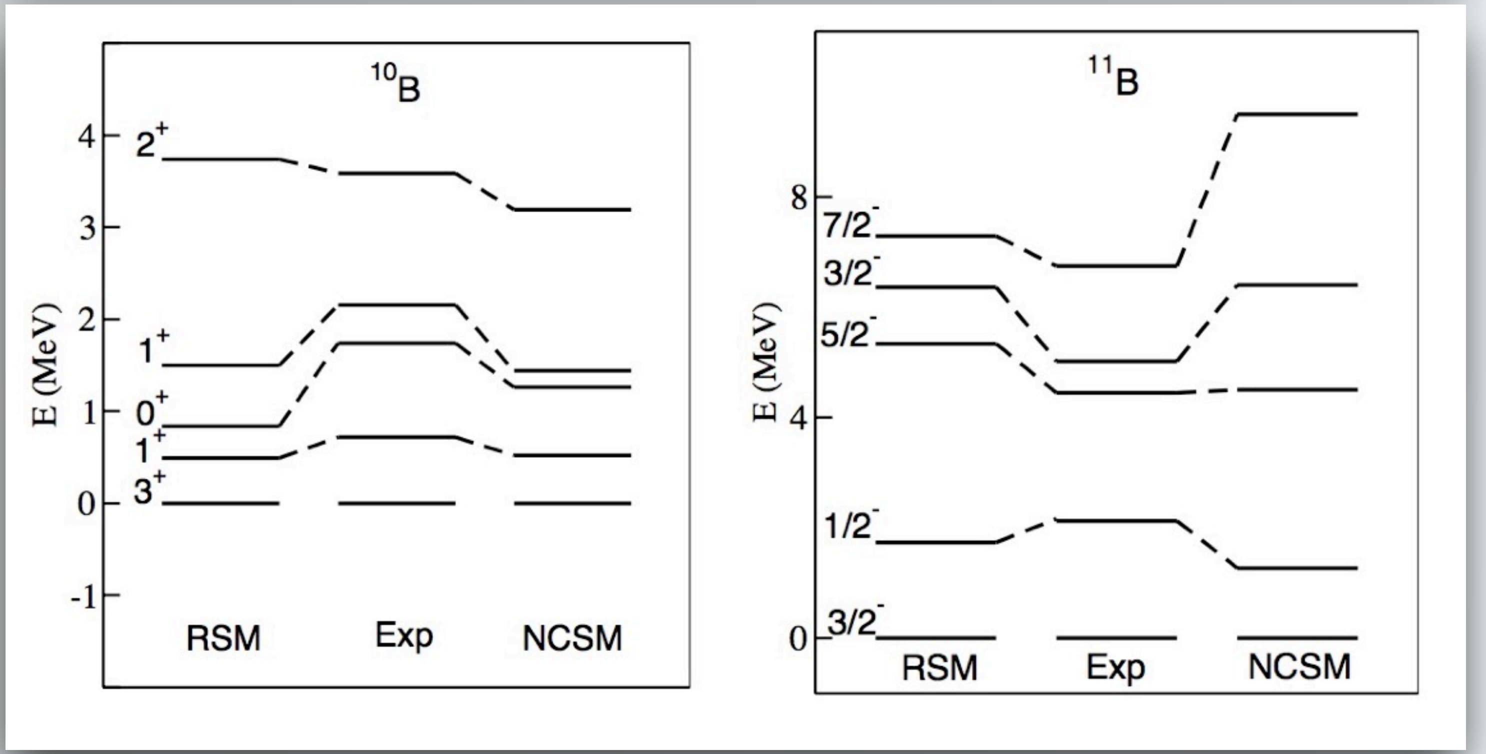
RSM calculation for p-shell nuclei

Benchmark with NCSM

***NN* potential only**



NN + 3N

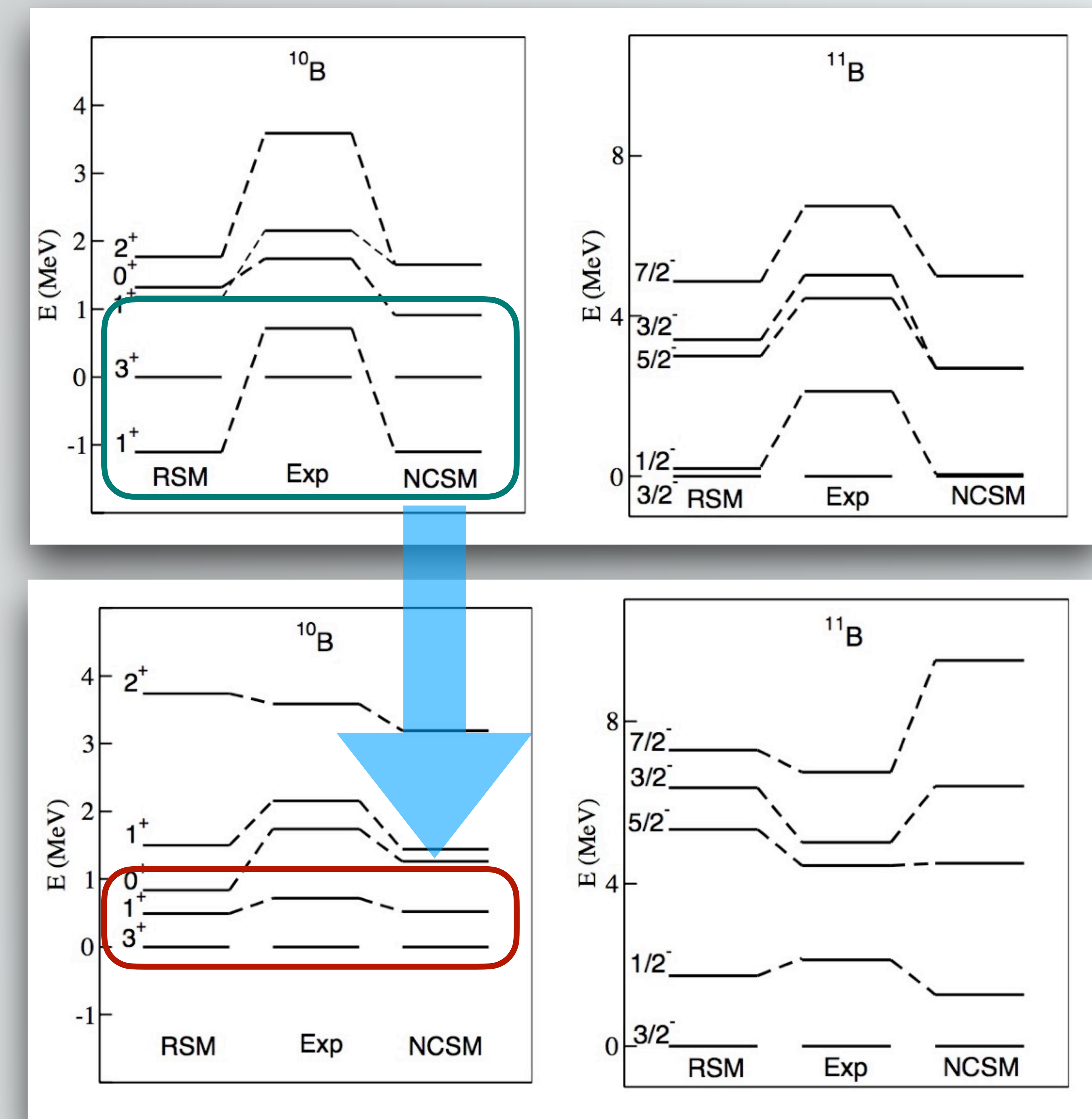


RSM: T. Fukui, L. De Angelis, **Y. Z. Ma** *et al.*, PRC **98**, 044305 (2018)

RSM calculation for p-shell nuclei

Benchmark with NCSM

***NN* potential only**



NN + 3N

RSM: T. Fukui, L. De Angelis, **Y. Z. Ma** *et al*., PRC **98**, 044305 (2018)

Outline

Chiral 3NF

Continuum effects

Chiral 3NF: origin, derivation, calculation & benchmark, implement (RSM or MBPT)

Continuum effect: Berggren basis, inclusion of 3NF, implement (GSM)

Outline

Chiral 3NF

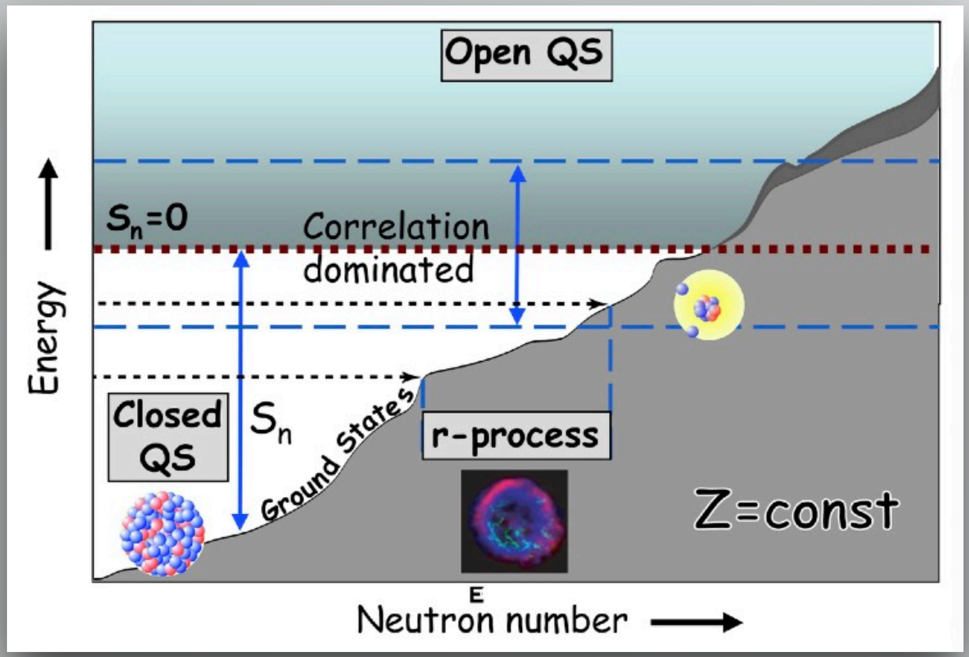
Weakly bound nuclear systems

Continuum effects

Chiral 3NF: origin, derivation, calculation & benchmark, implement (RSM or MBPT)

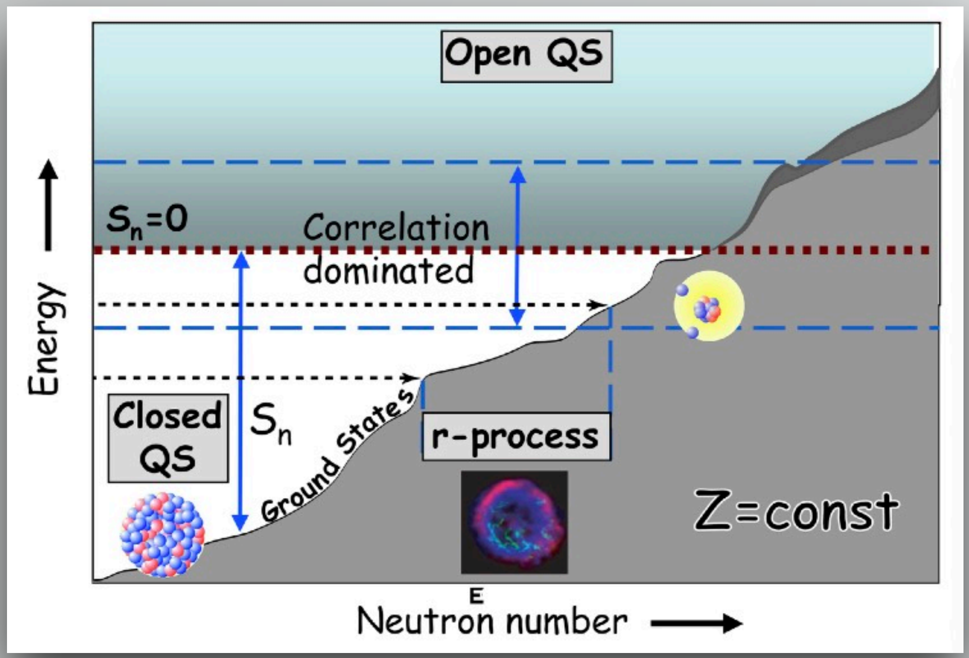
Continuum effect: Berggren basis, inclusion of 3NF, implement (GSM)

Gamow-Berggren basis (one-body)

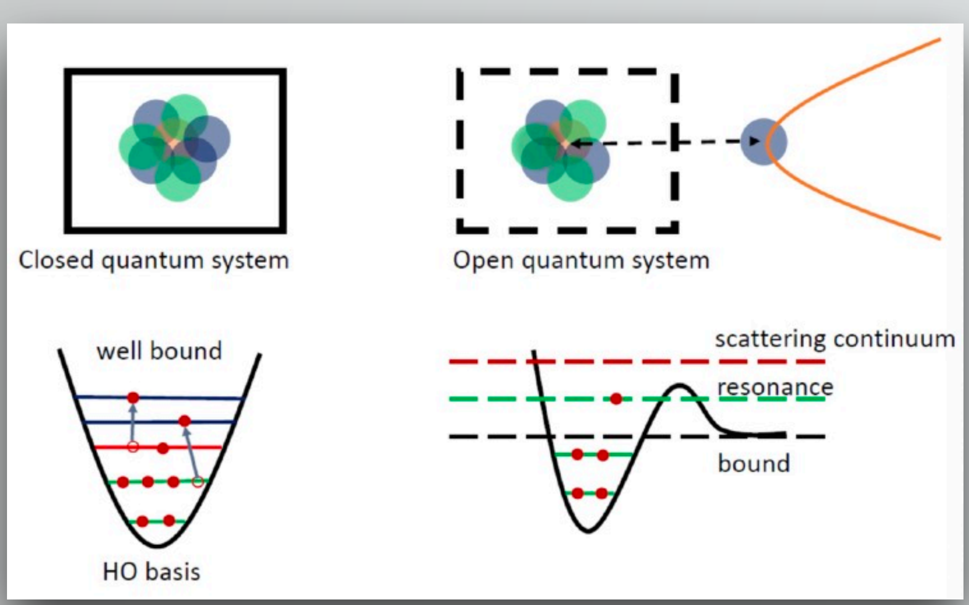


N Michel, W Nazarewicz, et al., J. Phys. G **36** (2009) 013101

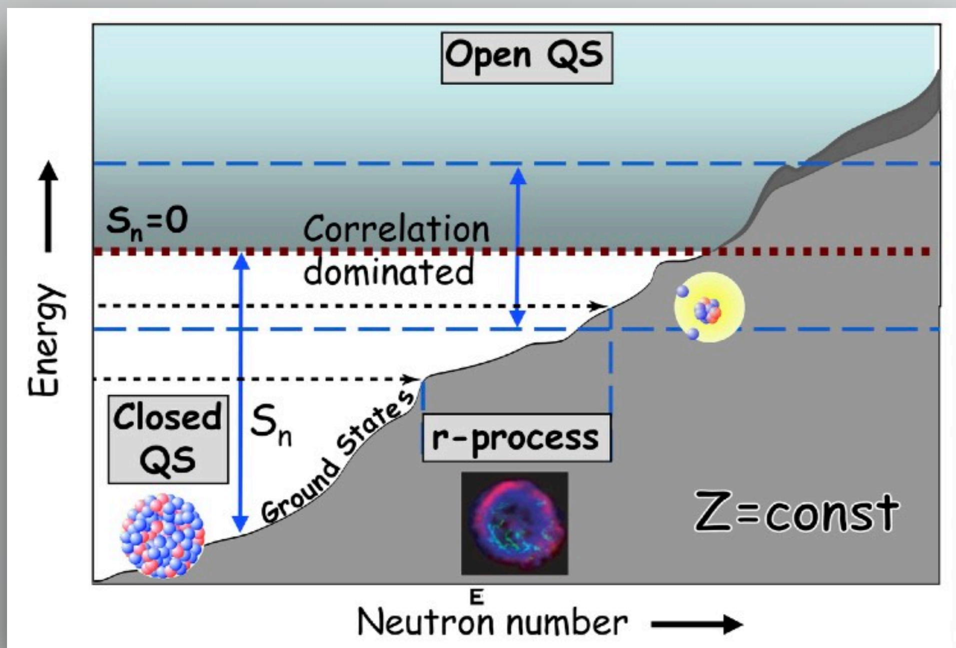
Gamow-Berggren basis (one-body)



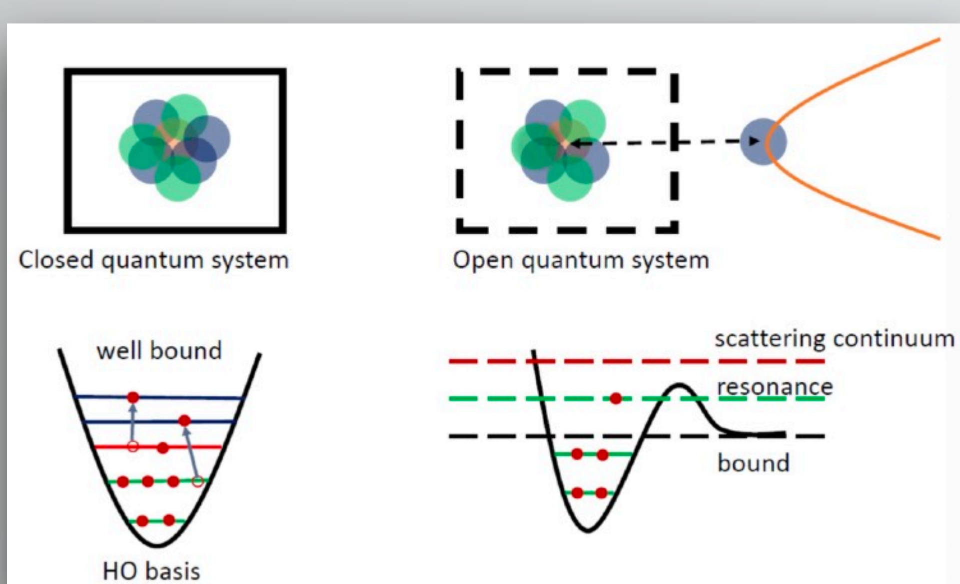
N Michel, W Nazarewicz, et al., J. Phys. G **36** (2009) 013101



Gamow-Berggren basis (one-body)



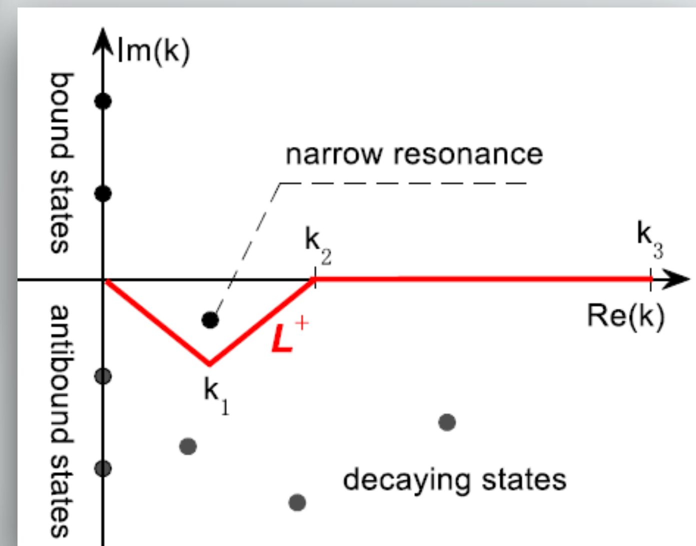
N Michel, W Nazarewicz, et al., J. Phys. G **36** (2009) 013101

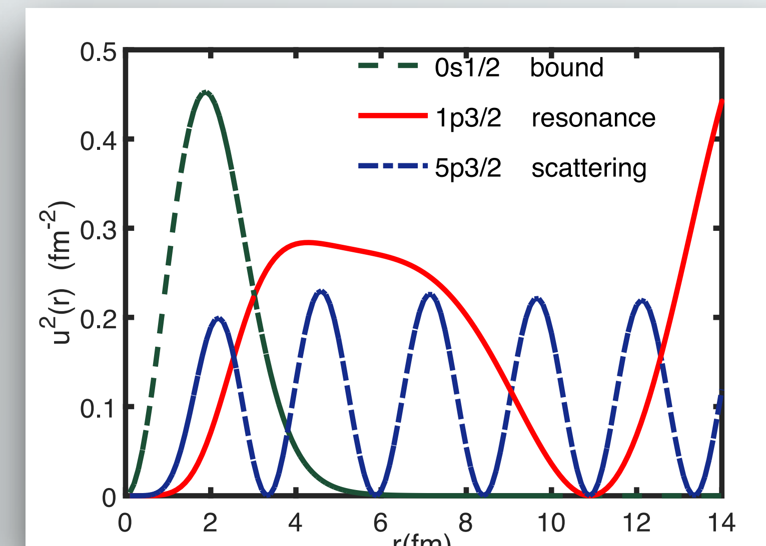


$$\frac{d^2u(k,r)}{dr^2} = \left[\frac{l(l+1)}{r^2} + \frac{2m}{\hbar^2}U(r) - k^2\right]u(k,r)$$ **in complex-*k* space**

$$e = \frac{\hbar^2 k^2}{2m} = e_n - i\frac{\gamma_n}{2}$$

Different wavefunction behavior of: bound, resonance, continuum states



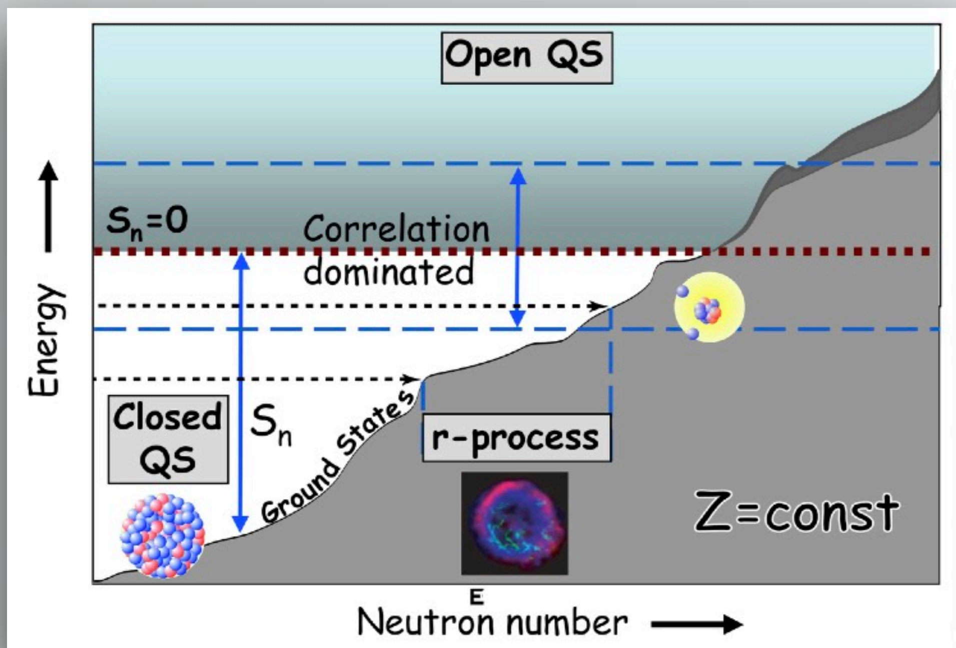


Z. H. Sun *et al.*, PLB 769 (2017) 227–232

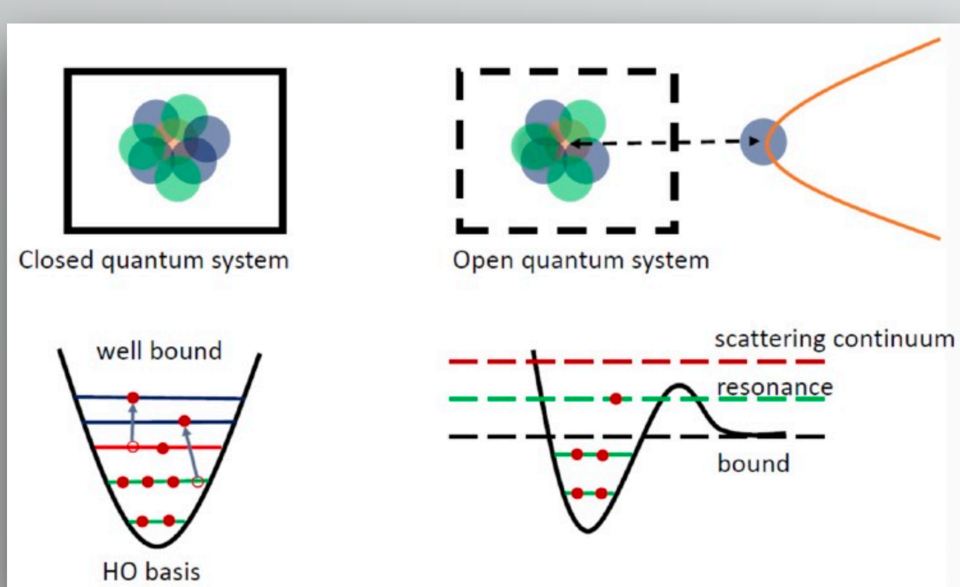
$$\sum_n u_n(r)u_n(r') + \int_{L^+} u(k,r)u(k,r')dk = \delta(r-r')$$

T. Berggren, Nucl. Phys. A109 (1968) 265

Gamow-Berggren basis (one-body)



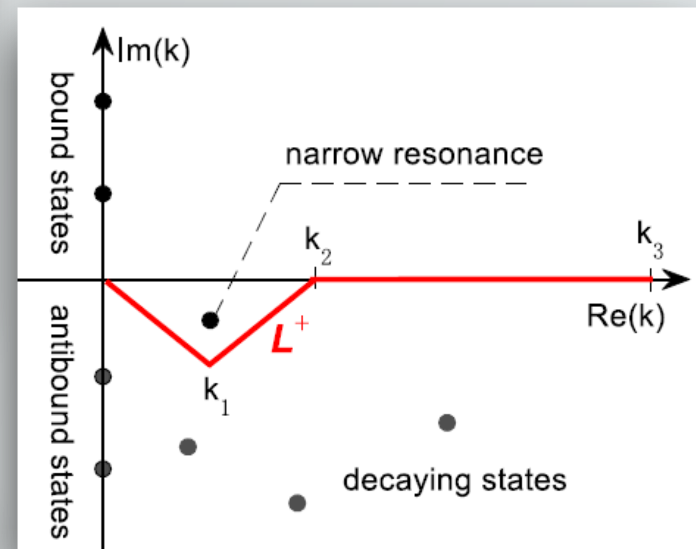
N Michel, W Nazarewicz, et al., J. Phys. G **36** (2009) 013101

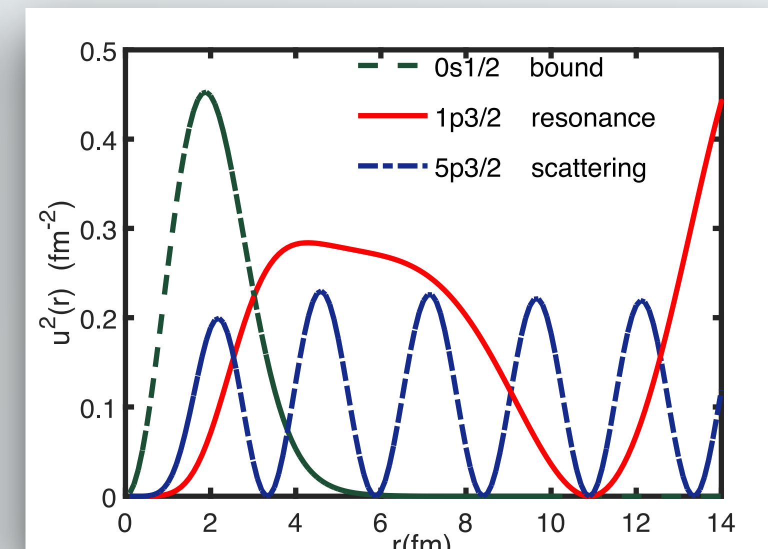


$$\frac{d^2u(k,r)}{dr^2} = \left[\frac{l(l+1)}{r^2} + \frac{2m}{\hbar^2}U(r) - k^2\right]u(k,r)$$ **in complex-*k* space**

$$e = \frac{\hbar^2 k^2}{2m} = e_n - i\frac{\gamma_n}{2}$$

Different wavefunction behavior of:
bound, resonance, continuum states





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$$\sum_n u_n(r)u_n(r') + \int_{L^+} u(k,r)u(k,r')dk = \delta(r-r')$$

T. Berggren, Nucl. Phys. A109 (1968) 265

Methods based on Berggren basis:

GSM: N. Michel, W. Nazarewicz, M. Płoszajczak, R. J. Liotta, …
GSM with realistic force: Z.H. Sun, B.S. Hu, Y. Z. M, F. R. Xu…
no-core GSM: G. Papadimitriou, N. Michel, …
Gamow CC: G. Hagen, D.J. Dean, M. Hjorth-Jensen, T. Papenbrock, …

Effective Hamiltonian (many-body)

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1. Add 3N contribution to Hamiltonian

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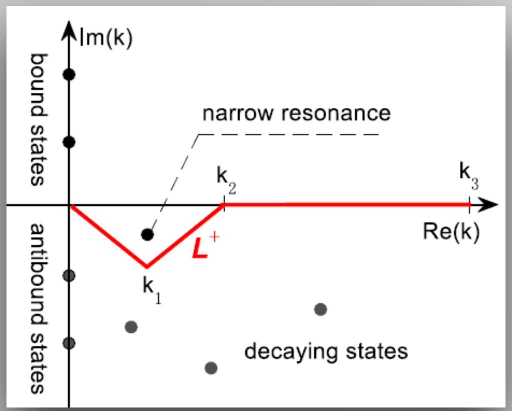
2. Transfer to Berggren basis



$$\langle ab|V|cd\rangle = \sum_{\alpha\leq\beta}\sum_{\gamma\leq\delta}\langle ab|\alpha\beta\rangle\langle\alpha\beta|V|\gamma\delta\rangle\langle\gamma\delta|cd\rangle$$

N_L=6+6+8

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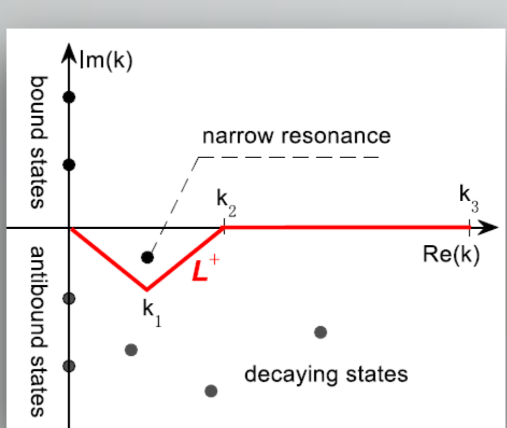

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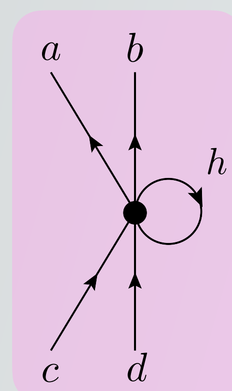

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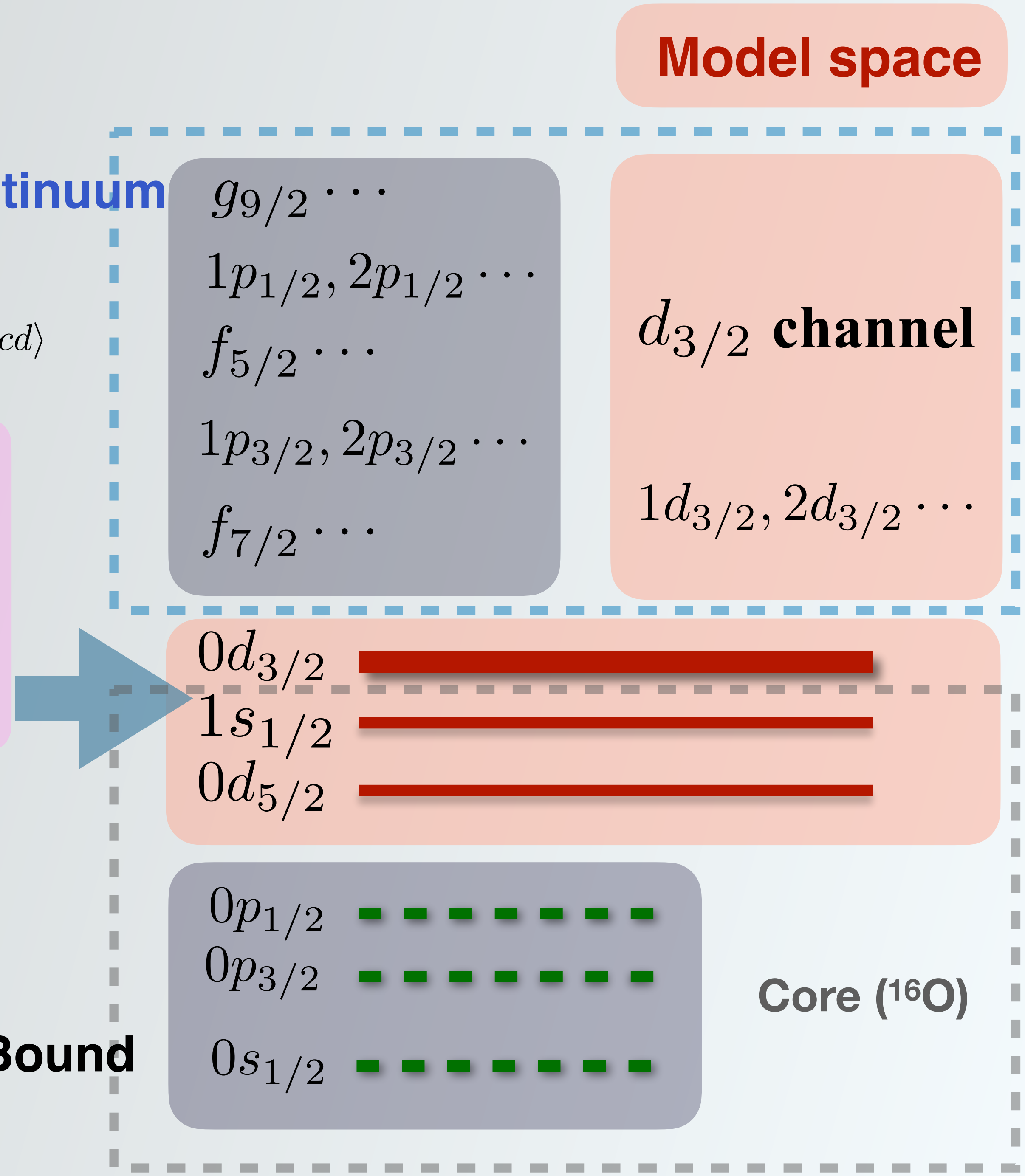
3. Calculate Q-box folded diagrams in complex-*k* space

$$\hat{Q}(E) = PVP + PVQ\frac{1}{E-H}QVP$$



Example: neutron rich Oxygen isotopes

Effective Hamiltonian (many-body)

1. Add 3N contribution to Hamiltonian

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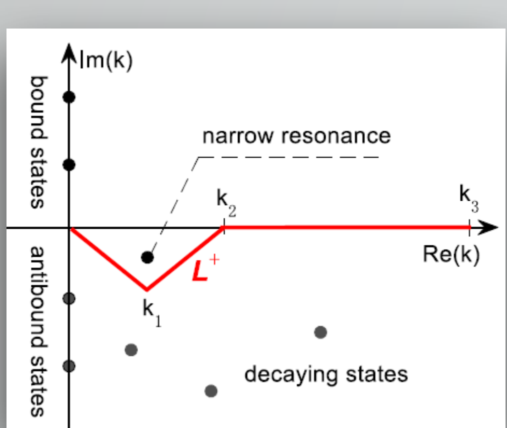

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Z. H. Sun *et al.*, PLB 769 (2017) 227–232

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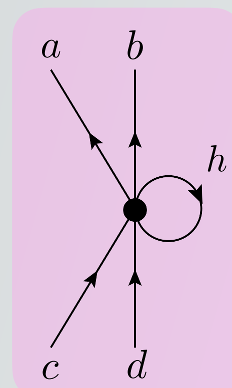
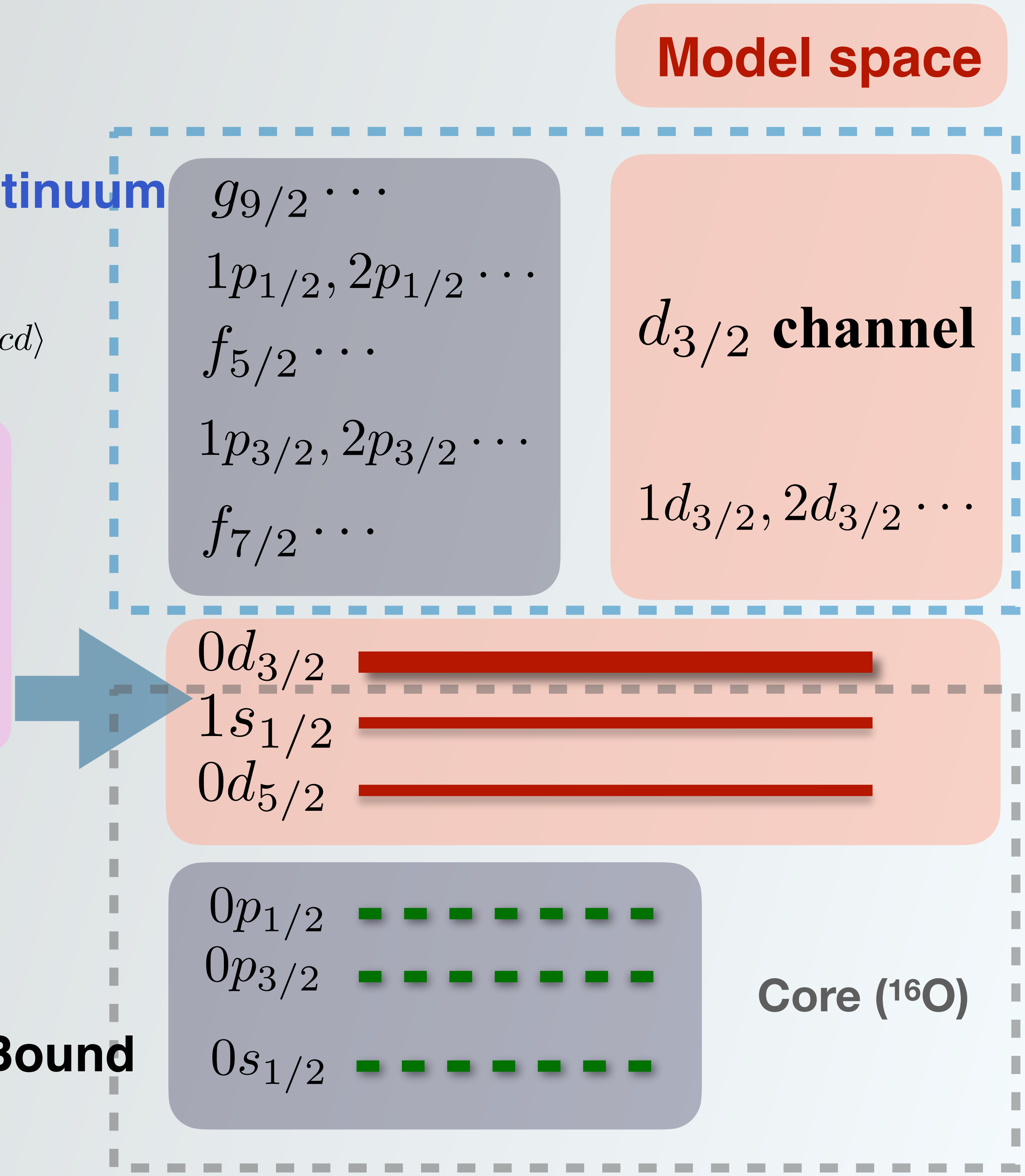




3. Calculate Q-box folded diagrams in complex-*k* space

$$\hat{Q}(E) = PVP + PVQ\frac{1}{E-H}QVP$$

4. Diagonalization in complex-*k* space by Jacobi-Davidson method (cooperation with Nicolas Michel)

Example: neutron rich Oxygen isotopes

Outline

Chiral 3NF

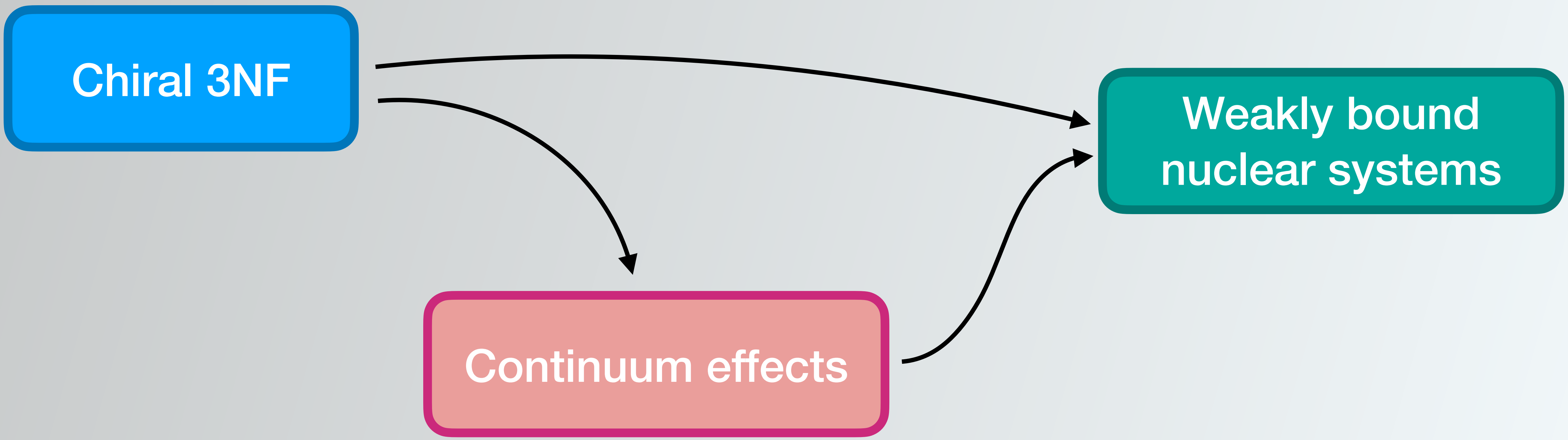
Weakly bound nuclear systems

Continuum effects

Chiral 3NF: origin, derivation, calculation & benchmark, implement (RSM or MBPT)

Continuum effect: Berggren basis, inclusion of 3NF, implement (GSM)

Outline



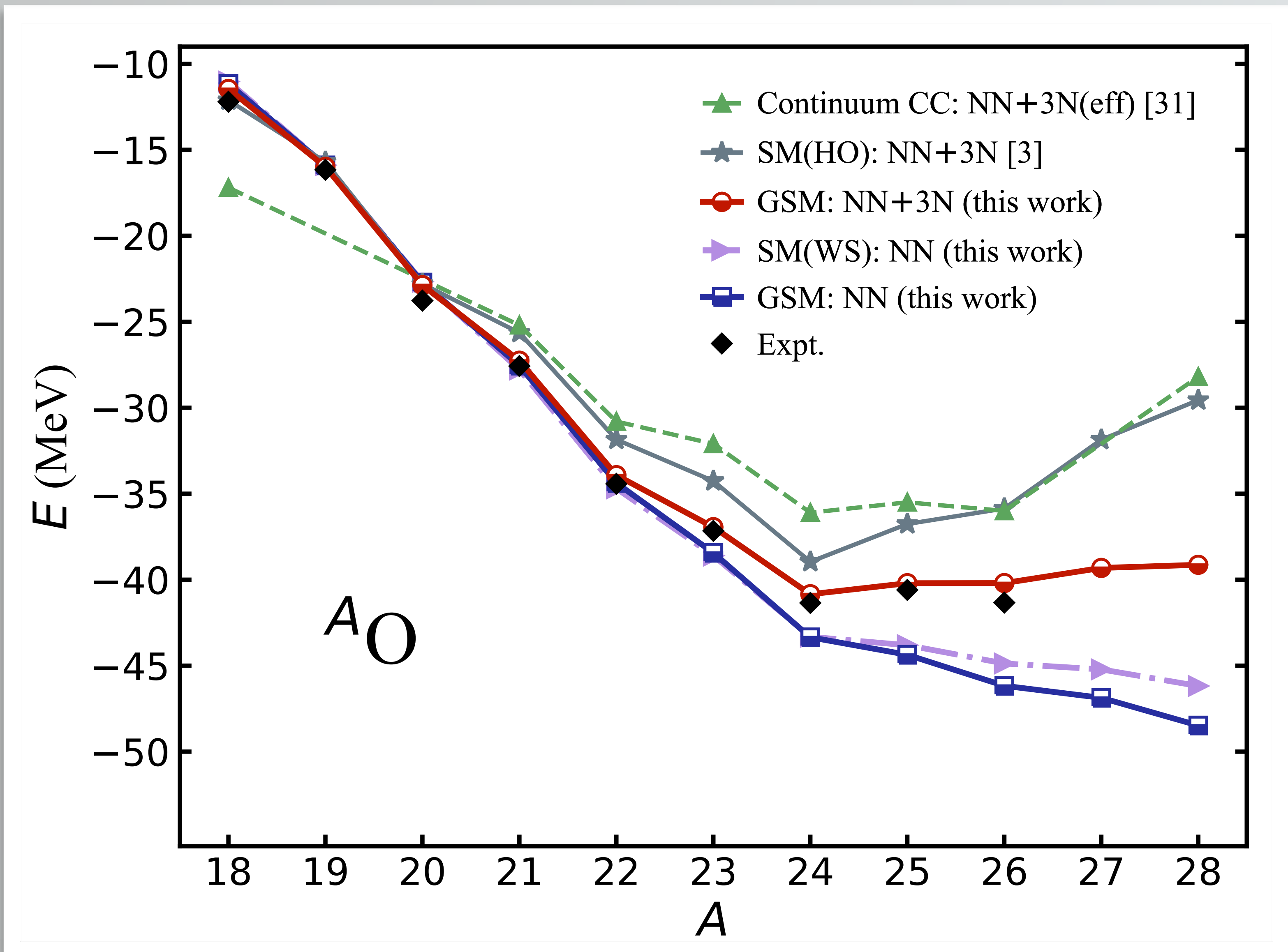
Chiral 3NF: origin, derivation, calculation & benchmark, implement (RSM or MBPT)

Continuum effect: Berggren basis, inclusion of 3NF, implement (GSM)

Weakly bound nuclear system:

1. neutron rich Oxygen isotopes
2. Borromean ^{17}Ne
3. Mirror symmetry breaking partners (张爽报告)

1) neutron rich Oxygen isotopes



NN: N^3LO two-body forces
3N: N^2LO three-body forces
(c_D=-1.0, c_E=-0.34)

S_{2n}(MeV)	NN	NN+3N	Expt.
^{24}O	9.110	6.924	6.925
^{25}O	6.254	3.259	3.453
^{26}O	3.362	-0.648	-0.018

- **3NF & Continuum is crucial to reproduce Oxygen drip line, especially for the ground state of ^{26}O.**
- **3NF behaves repulsive effects**
- **3NF effects increase rapidly as the increasing of neutron number**

NN + 3N: T. Otsuka *et al.*, PRL 105, 032501 (2010)
NN + 3N(Effective) + Continuum: G. Hagen *et al.*, PRL 108, 242501 (2012)

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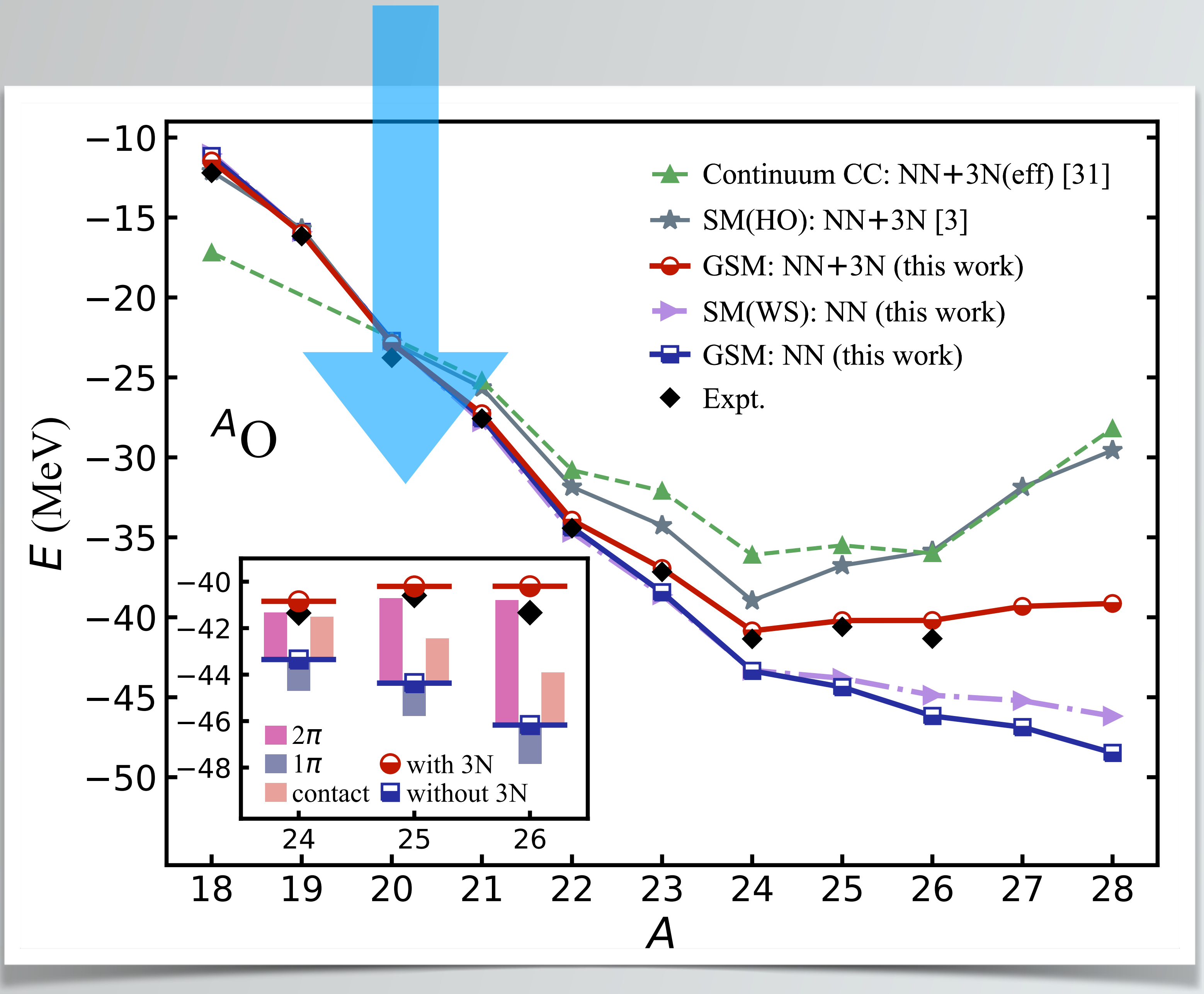
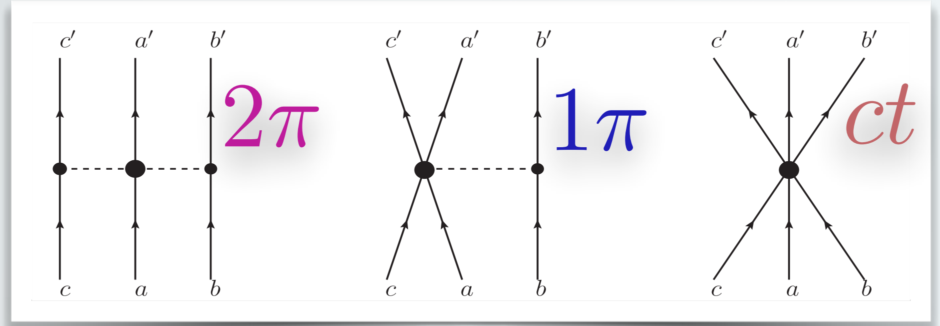
1) neutron rich Oxygen isotopes

Contributions from different components (2-π, 1-π and contact term) of 3NF





- Besides long-range 2-π exchange term,1-π exchange & contact term also have a significant contribution.
- 2-π exchange & contact term have repulsive effects, while 1-π exchange term has an attractive contribution.
- 1-π exchange + contact term has a small contribution.
- 2-π exchange term increases faster than the other two terms with the increasing of neutron number.

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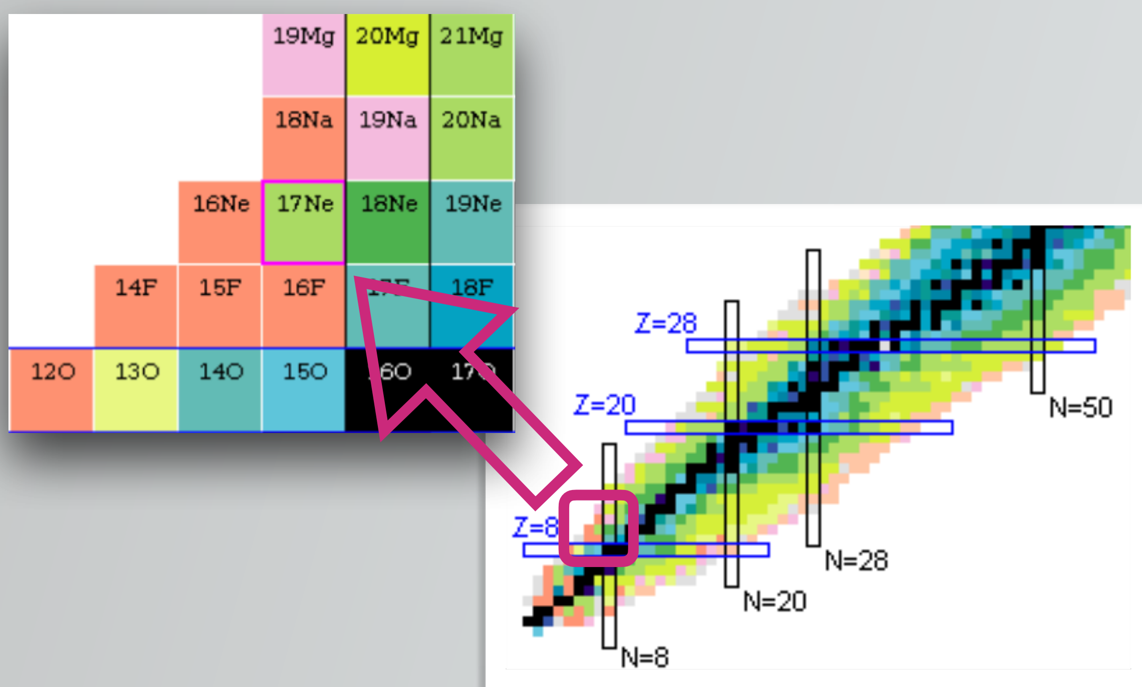
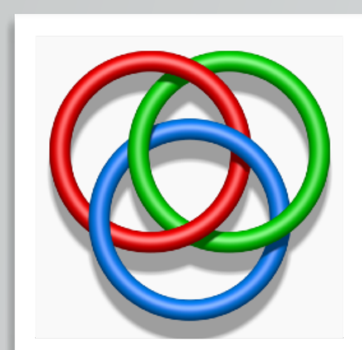
2) Borromean ^{17}Ne (3NF & continuum)





Borromean Ring

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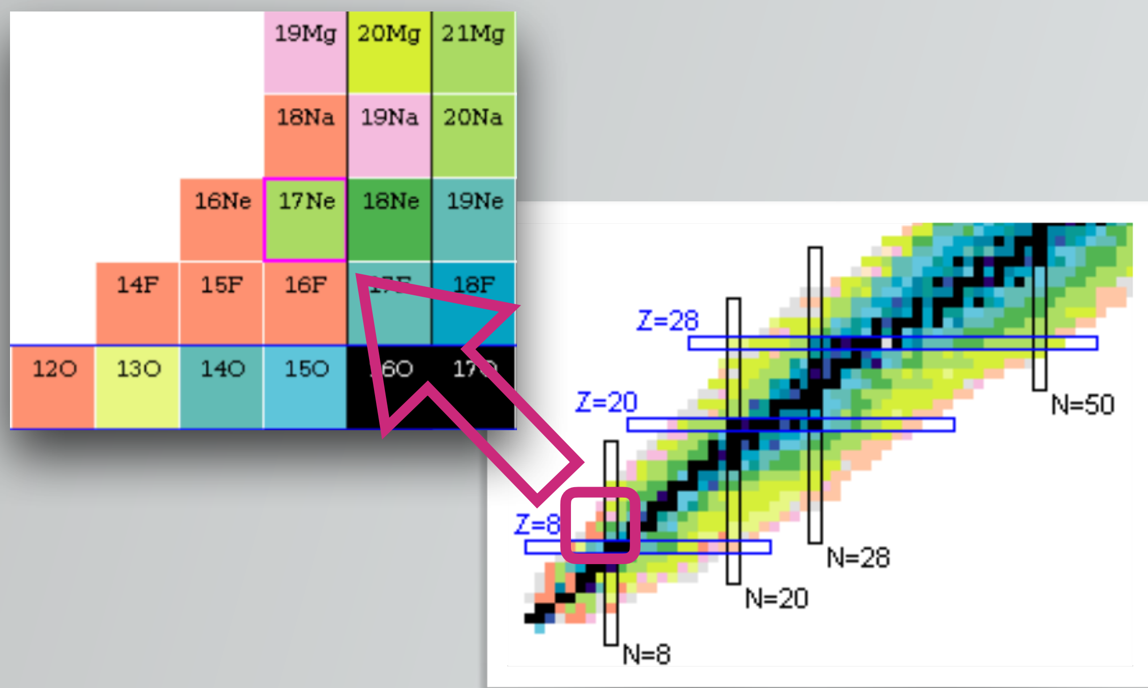
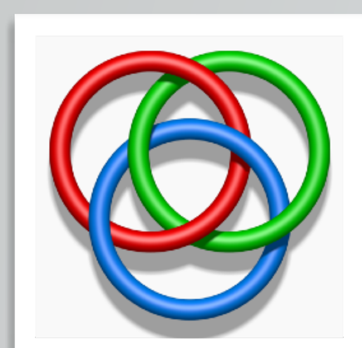
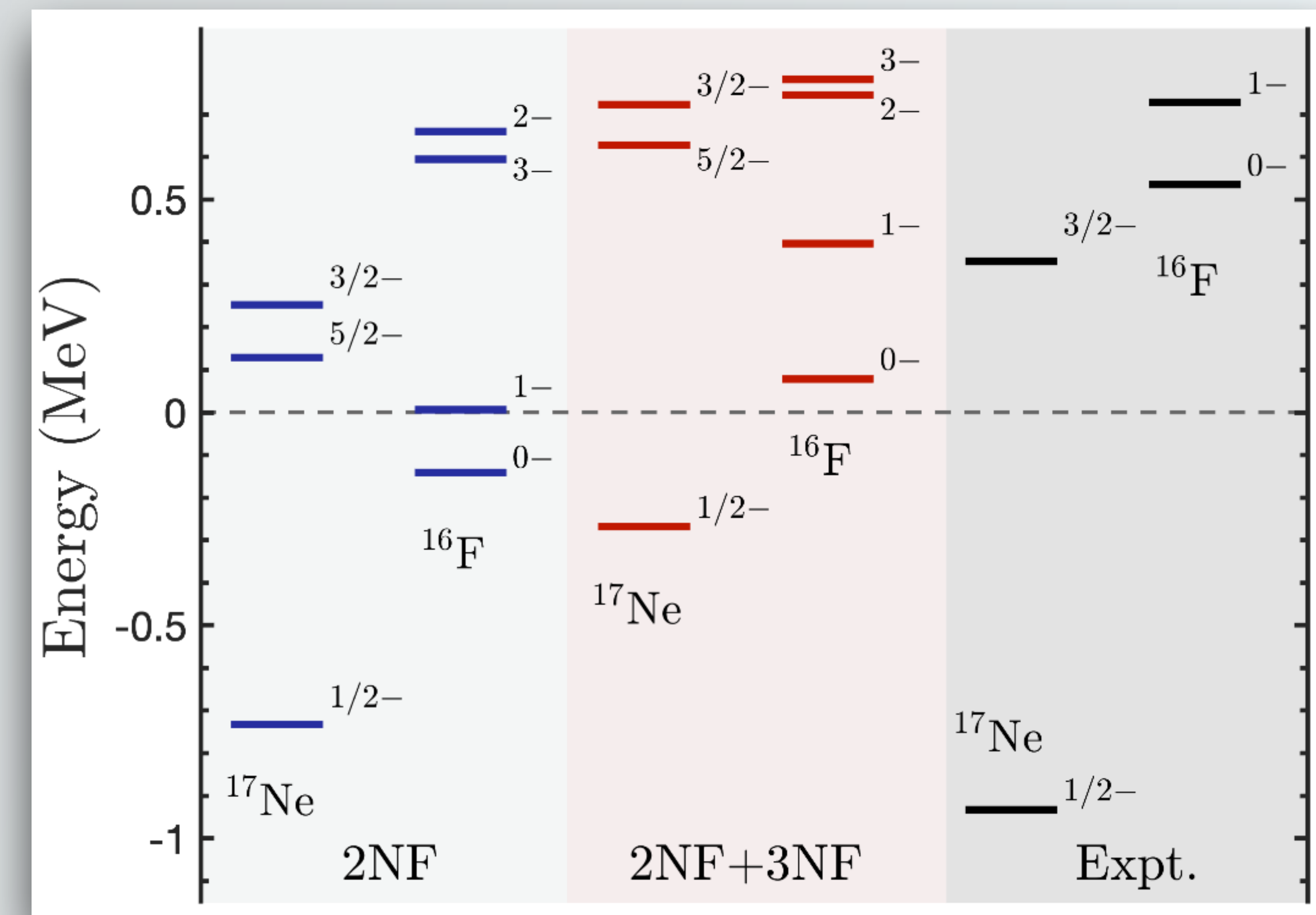




Borromean Ring

3NF is essential for the Borromean structure of ^{17}Ne



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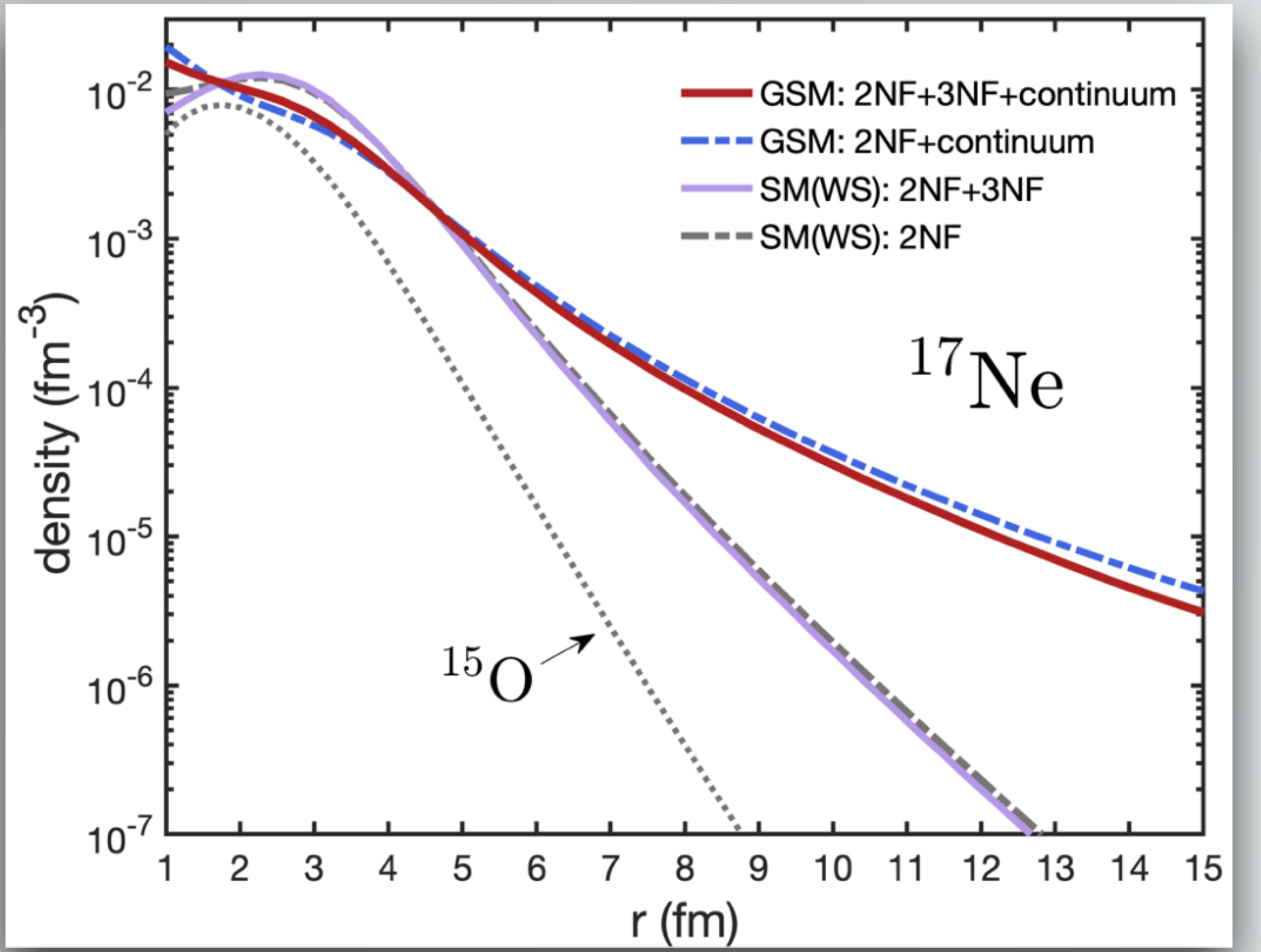
2) Borromean ^{17}Ne (3NF & continuum)

Continuum is more crucial for the Halo structure of ^{17}Ne



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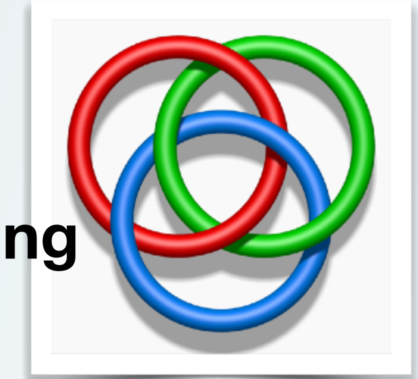
Borromean Ring



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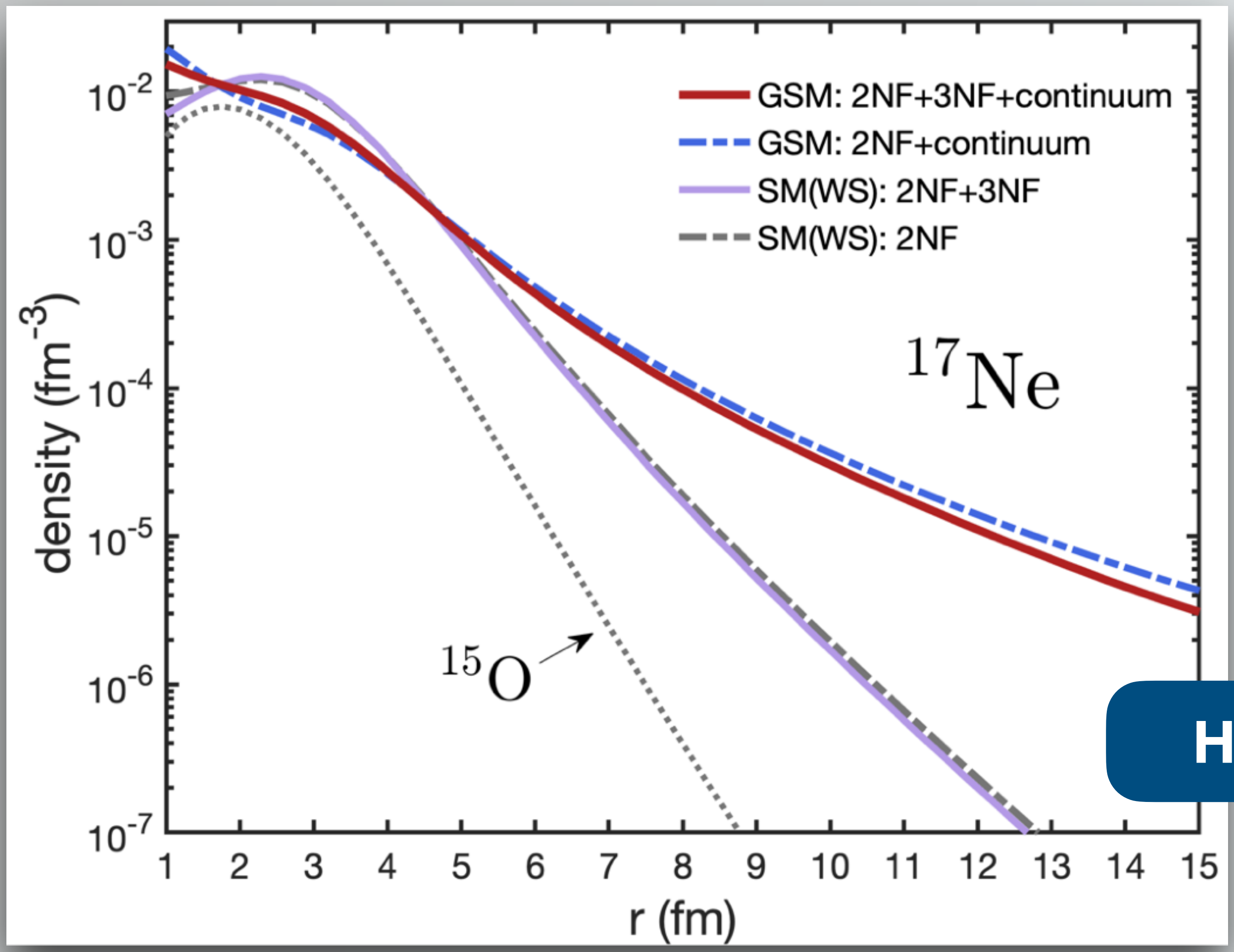

Halo structure

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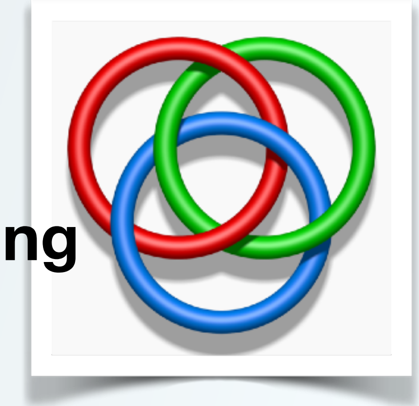
Borromean Ring



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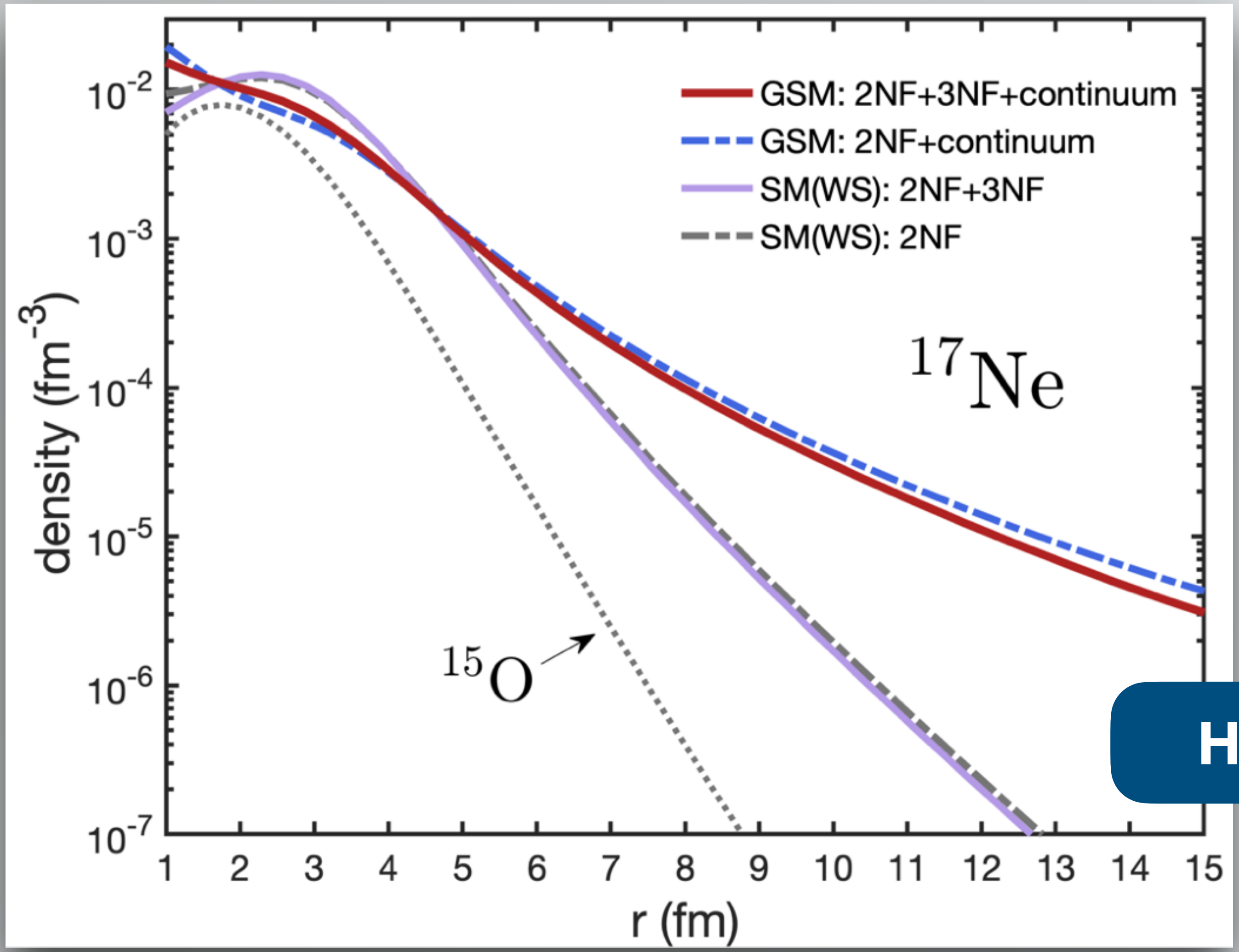
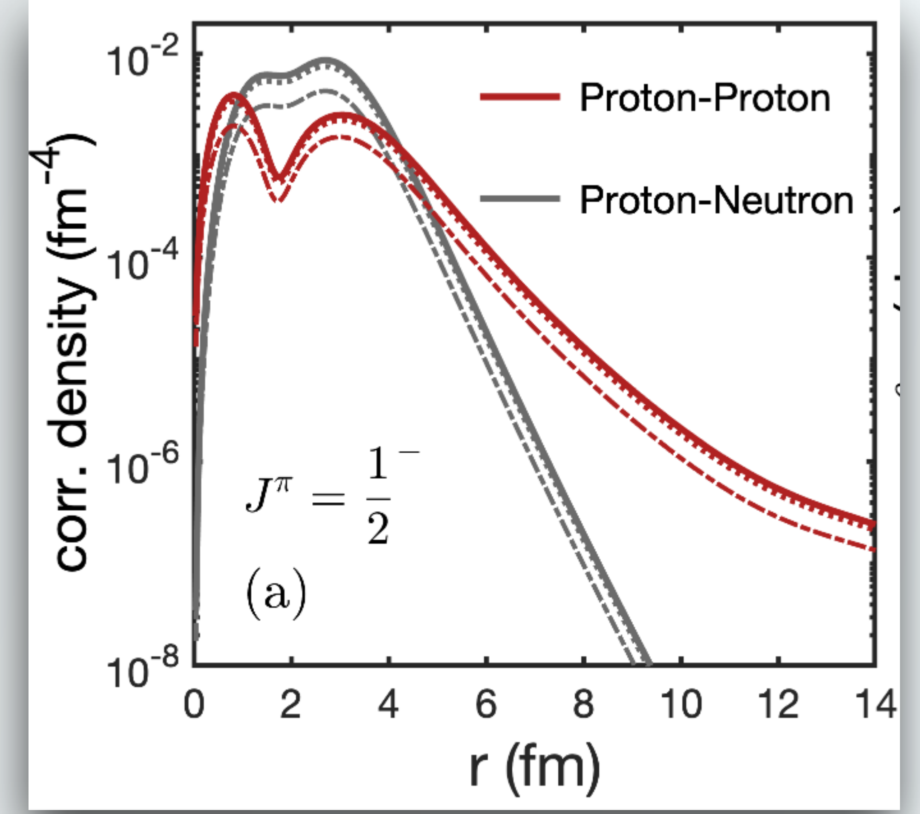
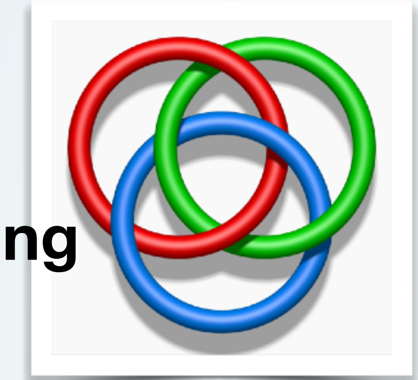


Correlation density



Halo structure

Borromean Ring



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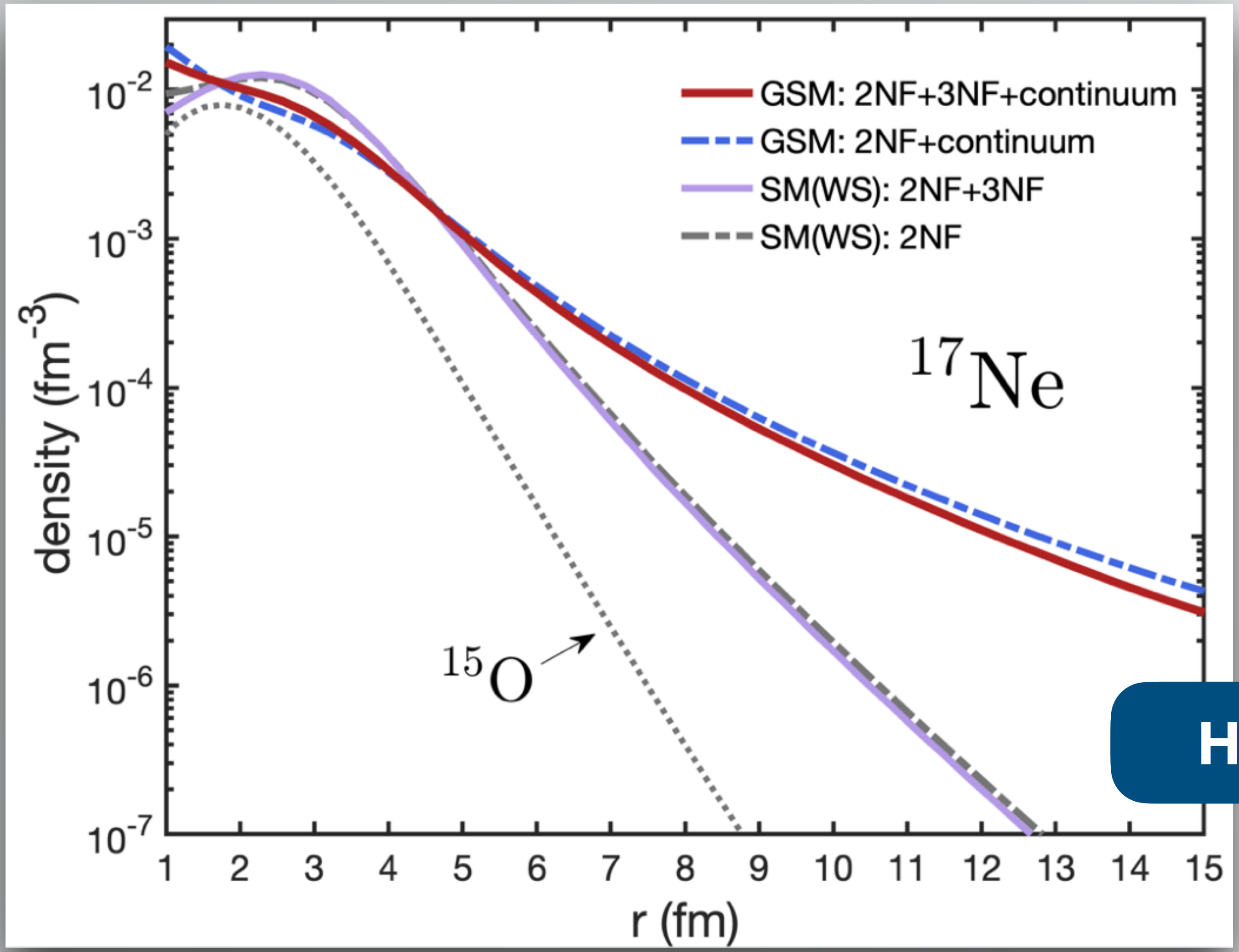

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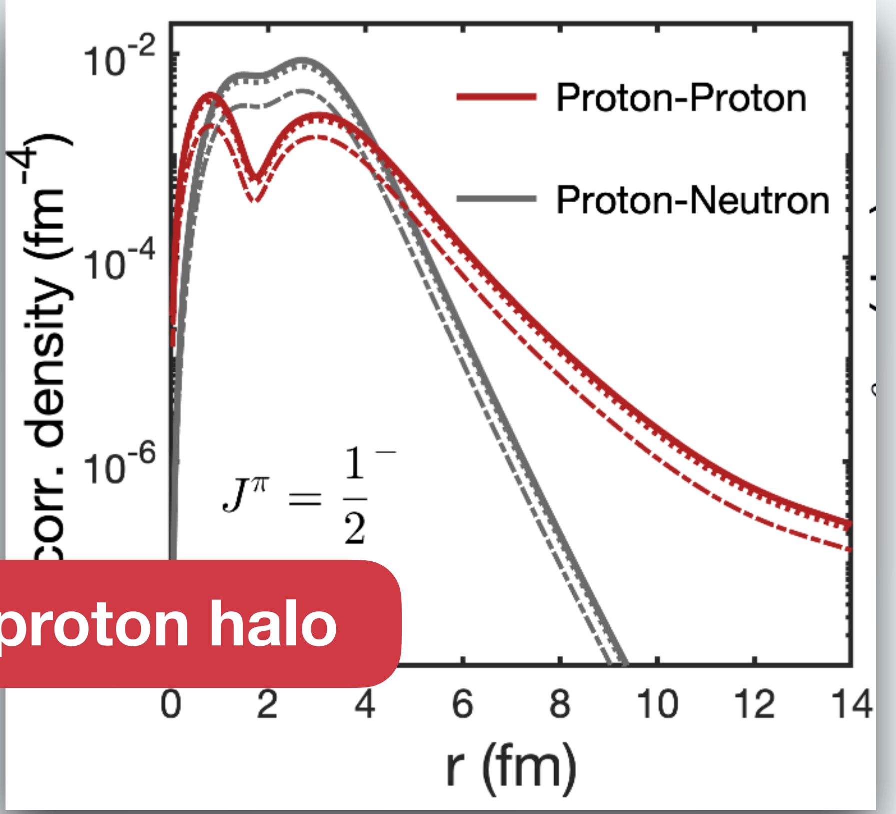
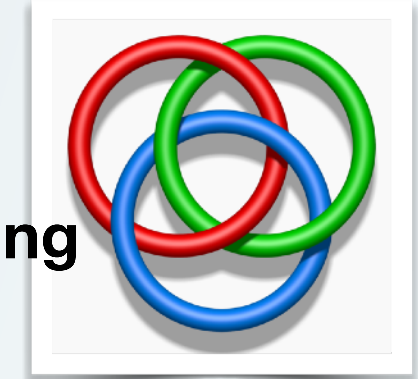


Two-proton halo

Halo structure

Borromean Ring



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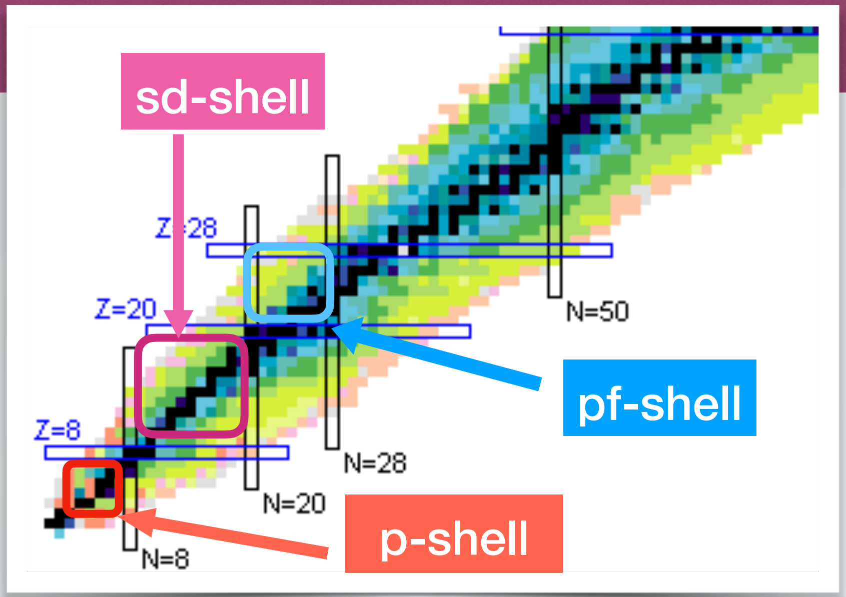
Summary (1-5):

1. Success to derive & calculate 3N matrix element from chiral NNLO

2. *p*-shell: test the reliability of our 3NF



3	4	5
fp-shell & shell evolution	*sd*-shell & neutron drip line	*sd*-shell & proton rich
RSM: ^{48}Ca -> ^{56}Ni	CGSM: ^{18}O -> ^{26}O	CGSM: ^{17}Ne Borromean

Gamow Basis (continuum effects)

Challenge:

- Including high order contribution from 3N force.
- To adopt 3N force to heavier nuclei we need calculate 3N matrix element in a much larger model space which means the demand of huge computation resource and highly optimized program.

谢谢！